SOME FLOW PROBLEMS OF NEWTONIAN AND NON-NEWTONIAN FLUIDS

A thesis submitted
In Partial Fulfilment of the requirements
for the Degree of

Doctor of Philosophy

by

Ramesh Chandra Srivastava M. Sc.

to the
Department of Mathematics
Indian Institute of Technology, Kanpur

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CERTIFICATE

This is to certify that the thesis ''Some Problems of Newtonian and Non-Newtonian Fluids'' by Sri Ramesh Chendra Srivastava, submitted in partial fulfilment of the Degree of Doctor of Philosophy of the Indian Institute of Technology, Kanpur is a record of bonafide research work carried out by him under my supervision and guidance, and that the results embodied in this thesis have not been submitted to any other University or Institute for the award of any Degree or Diploma.

J. N. Kapur

(J. N. Kapur)
Professor and Chairman
Mathematics Department
Indian Institute of Technology,
Kanpur.

Dated:
March 15, 1968.

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R.C. Srivastava

(R. C. Srivastava)

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L I S T O F T A B L E S

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LIST OF FIGURES

NOMENCLATURE

(x, y, z)	Cartesian Coordinates
(r, θ, z)	Cylindrical Coordinates
(r, ϕ, θ)	Spherical polar Coordinates
(u, v, w)	Velocity Components in the Coordinate directions
U, V, W	Potential Velocity Components
l, m, n	Integers or summation indices
$U(r, \phi)$	Velocity Components in r direction
$V(r, \phi)$	" " " ϕ "
$W(r, \phi)$	" " " θ "
e_{ij}	Strain-rate tensor
t_{ij}	Stress tensor
p_{ij}	Hydrostatic pressure
μ	Coefficient of Viscosity
ρ	Density of the fluid
ν	Kinematic coefficient of Viscosity
μ_c	Constant coefficient of cross-viscosity
μ, μ_a	Constants depending on the flow behaviour of the special type of non-Newtonian fluid in Chapter VI
η_0	Scalar Corresponding to Viscosity
n, K	Flow behaviour indices of the power-law fluid
$\psi(x, y)$	Stream function
L	Characteristic length

ξ	non-d parameter $\frac{x}{L}$ in Chapter II
	" " $\frac{r}{a}$ in Chapter VIII
η	" " $\frac{y R_1^{\frac{1}{n+1}}}{L g(x)}$ in Chapter II
	" " $\sqrt{\frac{a}{\nu}} \cdot \frac{y}{\delta}$ in Chapter IV
Υ	" " $\frac{r}{a}$ in Chapter VII and IX
ζ	" " $\frac{y}{h}$ in Chapter V
U_∞	Reference Velocity
$g(x)$	Suitable scaling function
$f(\xi, \eta)$	a function $\frac{\psi(x,y) R_1^{\frac{1}{n+1}}}{L U(x) g(x)}$
α	$\frac{L g^n(x) U_\infty^{n-2}}{n U_\infty^{n-1}} \frac{d}{dx}\left(U(x) g(x)\right)$
	otherwise an angle
β	$\frac{L g^{n+1}(x) U_\infty^{n-2}}{n U^{n-1}(x)} U'(x)$
R	Reynolds number $\frac{Uh}{\mu/\rho}$ or $\frac{\rho A a^3}{4\mu^2}$
R_1	Modified Reynolds number $\frac{L}{\nu U_\infty^{n-2}}$
$B_x(p,q)$	Incomplete β-function
'a'	Undetermined Coefficient of the Velocity profiles in Chapter III
σ	Ratio $\frac{b}{a}$ of the radii of the annulus

λ	Relaxation time parameter in Chapter VII and IX
τ	Retardation time parameter
	Shear stress in Chapter I
V	Velocity of suction and injection
$2D$	Diameter of the pipe in Chapter III
δ	Boundary layer thickness
R	Ratio of the integrated kinetic energies
$\bar{\Phi}$	Viscous dissipation function
Q	Flow rate of the fluid through channel
K_o	Non-d parameter $\left(\frac{R\bar{P}}{R_1}\right)^{1/n}$ in Chapter VI
l	Radius of curvature of the curved channels
T or T_l	The temperature inside the curved channel
t_{1l}	Temperature inside a straight channel
t_l	Perturbation temperature inside the curved channel
t_w	Wall temperature
Δ	Operator $\frac{\partial^2}{\partial \xi^2}+\frac{1}{\xi}\frac{\partial}{\partial \xi}-\frac{1}{\xi^2}$
C_p	Specific heat at constant pressure
k	Thermal conductivity in Chapters VII to IX
P_r	Prandtl number
$\frac{D}{Dt'}$	Material derivative

$\Phi = \pm \frac{\pi}{2}$	Convex or Concave side of the horizontal diameter of the curved channels
D	$\frac{1}{4 \ln \sigma} \left[2\lambda^2 (\ln \sigma)^2 - 4\lambda (1-\sigma^2) - (1-\sigma^4) \right]$
	in Chapters VII and IX
S	Non-d parameter $P_r . R^2$
ln	Natural logarithm
w_1	Velocity field in a straight channel
w	Perturbation velocity in a curved channel

SYNOPSIS

Ramesh Chandra Srivastava, M.Sc.
Ph.D.
Indian Institute of Technology, Kanpur, INDIA
March 1968
Some Flow Problems of Newtonian And Non-Newtonian Fluids

The present thesis consists of the following two parts :

Part I : Flow problems of non-Newtonian (power-law and visco-elastic)fluids.

Part II : Flow and heat-transfer problem of Newtonian fluid in curved channels.

In the first part (chapters II to VI) we are concerned with the motion of three types of non-Newtonian fluids viz.

(i) Power-law fluids,

(ii) Special type of non-Newtonian fluid exhibiting both power-law and Newtonian behaviour,

(iii) Oldroyd's model of visco-elastic fluid.

In the second part (chapters VII to IX), we discuss the stream line flow of a Newtonian fluid through a curved annulus and heat-transfer in both a curved pipe and a curved annulus.

A short account of the developments of the research on general fluid flow problems of non-Newtonian fluids and

of the flow problems of Newtonian fluids in curved channels has been given in the introductory first chapter of this thesis. Apart from the general survey of the allied problems, this chapter also contains the discussion of these flow problems that have led us to formulate our own research problems. This chapter also contains the discussion on the constitutive equations of those fluids which we have used in our research problems.

In the second chapter we have developed the theory of ''Similar solutions of the boundary layer equations for power-law fluids'' for both the pseudo-plastic and dilatant models. We obtain in this manner the generalisations of the Falker-Skan equation. The velocity distribution of the potential flow is found to be proportional to a power of the length of arc measured along the wall from the stagnation point. Important particular cases that we have considered here include the following :

(i) the boundary-layer flows along a wedge,

(ii) the boundary-layer flows along a flat plate,

(iii) the boundary-layer flows in a convergent channel,

(iv) two-dimensional stagnation point flows.

In the third chapter we have discussed the pressure drop in the inlet length of a circular pipe for the flow of a non-Newtonian power-law fluid. Following the Karman-Pohlhausen method, by assuming a cubic and two fourth degree velocity profiles, all the constants occuring in the

velocity profile equations can be determined from the boundary conditions except the one ''a''. To determine this constant Bogue (1959), on kinetic energy considerations, had suggested this ''a'' to be equal to $\frac{n+1}{n}$ where n is the flow behaviour index of the power-law model. We have suggested and used an alternative principle viz. a combination of the principle of the least squares and the principle of equal integrated kinetic energy to evaluate this constant, and have compared our results with those of Bogue (1959).

In the fourth chapter we have used a sixth degree velocity profile using Kármán-Pohlhausen method for the study of the axially-symmetric flow of an Oldroyd model of single-relaxation-time visco-elastic liquid near a stagnation point. It is found that both the boundary-layer thickness and the skin friction decrease with the increase of non-dimensional visco-elastic parameter λ. The two-dimensional stagnation point flow has also been considered for the same model of visco-elastic fluid and here we find that the boundary layer thickness steadily increases as the visco-elastic parameter λ increases. The dissipation function for the same fluid has also been found to agree with the earlier standard result.

In the fifth chapter, the steady flows of a visco-elastic fluid exhibiting only the relaxation time phenomena, in an annulus and between two parallel plates have been considered. It is assumed that the fluid is injected at one of the boundaries at the same rate as it is withdrawn

at the other and one of the plates (or cylinders) is moving parallel to itself with a constant velocity. We have also found the solutions for the case when the velocity of injection and suction has any value but the relaxation time parameter is small and vice-versa.

In the sixth chapter, we have obtained the exact analytical solutions for the flow of a non-Newtonian fluid characterised by the rheological equation :

$$t_{ij} = \mu e_{ij} + \mu_a' \left(\sum_{l=1}^{3} \sum_{m=1}^{3} e_{lm} e_{ml} \right)^{\frac{n-1}{2}} e_{ij}$$

both between two parallel plates and in a circular pipe under a constant pressure gradient. The solutions for the general value of n being very complicated algebraically, we have also given the series solutions for the fluids which either the Newtonian or the power-law term may be regarded as a perturbation over the other more dominant term. We have also obtained the solutions for some particular values of n .

In the seventh chapter, we have carried out theoretical calculations for the steady, laminar, incompressible flow of an ordinary viscous Newtonian fluid in a curved annulus. Corresponding to stream function ψ , pressure gradient p and the velocity w' for the case of a straight annulus we have found following Dean (1927) ψ', p' and w' respectively for the curved annulus on the supposition of secondary flows

developed in the flow region due to the curvature which has been assumed to be small. At first the flow parameters are evaluated for the general case and then a particular example is solved numerically. The graphical representations of the flow line in the plane of symmetry and the projection of stream lines on a normal cross-section are given. Only the case of large radius of curvature of the annulus is considered; more precisely this mean that the ratio of the radius of the outer curved pipe to that of the circle in which the common axis of the two curved pipes is coiled is sufficiently small.

In the last two chapters (chapter VIII and IX), we shall be presenting our theoretical studies in the field of convective heat transfer problem in curved channels. These curved channels are : (i) curved pipe with circular cross-section (ii) curved annular space between two circular cylinders.

In the eighth chapter we have considered the convective heat-transfer flow of incompressible Newtonian liquid under a constant pressure gradient through a circular pipe whose axis forms a circle of large radius ℓ in comparison with the radius of the cross-section. The temperature of the wall is kept constant both with respect to time as well as with respect to direction, and both the velocity and the temperature profiles are fully developed. We have adopted the method of imposing a perturbation temperature over the known results for the corresponding heat-transfer problem for

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a straight circular pipe. A number of graphs have been drawn to show the temperature profile inside the pipe and to show the local heat transfer rate on the wall of the pipe giving the effect of the curvature of the pipe. We have also calculated the skin friction on both the convex and the concave side of the pipe and found out that the higher temperature difference in the liquid on both sides of the vertical diameter is more due to the value of the Prandtl number P_r than due to the rise in skin friction, because the skin friction is higher on the convex side of the pipe than on the concave side.

In the ninth chapter we have considered a problem similar to that of the eighth chapter i.e. here the curved channel is the annular space between two concentric circles rotated round an axis in the same plane distant ℓ from the centre. We have considered the fully developed (both hydrodynamically and thermally) laminar forced flow. The geometry of the annulus is the same as that in chapter VII, and the method of approach to the problem, the same as that in the preceding chapter. Both the walls of the annulus (for the sake of simplicity) are kept at the same temperature. We have been able to evaluate the temperature profile through the annular region and also the local heat transfer rate at the inner and the outer walls of the annulus.

The work presented in this thesis forms part of the following papers :

1. Similar Solutions of Boundary Layer Equation for Power-law Fluids
ZAMP Vol. 14, Fasc. 4, 383 - 389 (1963).

2. Similar Solutions of Boundary Layer Equation for Power-law Fluids
A.I.Ch.E. Journ. 10, 776 (1964).

3. Boundary layer Velocity Profiles for the Flow of a Power-law Fluid in the Inlet Length of a Circular Pipe
Maths. Semi. III, 21 - 27 (1963).

4. Axially Symmetric and Two-dimensional Stagnation Point Flows of a Certain Visco-elastic Fluid
Jour. Phys. Soc. of Japan, Vol. 18, 3, 441 - 444 (1963).

5. On steady Motion of a Visco-elastic Liquid Between two Plane Boundaries or in an annulus with and without Suction
Journ. Engg. and Science Vol. VII, I, 127 - 142 (1963).

6. A Note on Impossibility of Some flows for General Reiner-Rivlin Fluids
Math. Seminar. Vol IV No1. (1966) 38-45

7. Hydrodynamic lubrication of an externally Pressurised Bearing Using a Reiner-Rivlin Fluid as Lubricant
App. Sci. Res. VOL. 14 - A, 133 - 137 (1964).

8. Stream line flow Through a Curved Annulus
App. Sci. Res. Vol. 14 Section A, 253 - 267 (1964).

P A R T   O N E

CHAPTER I

INTRODUCTION

1.1 INTRODUCTION :

The present chapter aims to give brief account of the problems of Non-Newtonian fluid flows and of Newtonian fluid flows in curved channels, so as to place our own contributions to these problems in their proper perspective.

Section 1.2 discusses briefly what non-Newtonian fluids are while section 1.3 gives some of the constitutive equations for these fluids, specially for those fluids with which we are going to be concerned in part I of this thesis. Section 1.4 gives an account of the work done on boundary layer flows of power-law fluids and of our own contributions to the same which are presented in detail in chapters II, III

and IV of this thesis. Section 1.5 gives a corresponding account for visco-elastic fluid flows which we have discussed in chapter V of this thesis. Section 1.6 gives a brief account of our investigations on non-Newtonian fluids which are not included in the main thesis. Sections 1.7 and 1.8 give a survey of the experimental and theoretical studies on fluid flow and heat transfer in curved channels and give the main results obtained by us.

1.2 NON-NEWTONIAN FLUIDS

The rheological equation of state (or the constitutive equation) of a body is defined in general as the relation between kinematic and dynamical variables for the body. For simple one-dimensional flow of the simplest type of fluid commonly known as Newtonian fluid this is defined by a direct proportionality between shear-stress τ and the induced rate of shear ($\partial u/\partial y$) so that

$$\tau = \mu \frac{\partial u}{\partial y} \qquad (1.2.1)$$

The constant of proportionality, μ is called the fluid viscosity. While many of the commonly known fluids are Newtonian, yet there are many others which deviate from this model. All the fluids that donot conform to the simple law (1.2.1) are classified as non-Newtonian; obviously, from mathematical point of view, the definition of non-Newtonian behaviour includes a far wider class of rheological relations

than that of Newtonian fluid. Commonly known non-Newtonian fluids are the slurries, soap solutions, paster, tars, various suspensions, high polymers, greases, mineral oils, paints and varnishes and a large number of lubricating oils etc.

Non-Newtonian fluids may be classified roughly into the following three classes according as the non-Newtonian viscosity η depends only on the state of shear or it depends on the previous history of the flow as well as on the state of shear or that the fluid exibits both the viscous and the elastic features.

1. Purely Viscous Fluids
2. Time-dependent Fluids
3. Visco-elastic Fluids.

We shall in this thesis, discuss the flow behaviour of the first and the third types of the non-Newtonian fluids only.

PURELY VISCOUS FLUIDS

Purely viscous fluids are those for which the shearing stress is a function of rate of strain only. This class of fluids may further be subdivided into :

(a) Bingham Plastic Fluids

(b) Pseudo-plastic Fluids

(c) Dilatant Fluids

depending on the nature of relationship of shear-stress and the rate of shear. Bingham plastic fluid is the simplest type of

non-Newtonian fluids in the sense that the relationship between the shear stress and the shear-rate differs from that of a Newtonian fluid only by the fact that, for one-dimensional flow, though the relationship is linear, its graph does not pass through the origin. Thus a finite shearing stress is necessary to initiate the movement. The plastic flow of an isotropic fluid is described by Oldroyd (1950) by the following system of equations :

$$
\begin{aligned}
t_{ii} &= 0 \\
e_{ii} &= 0 \\
e_{ik} &= \frac{1}{2\eta}\, t_{ik} \\
\frac{1}{\eta} &= \frac{1}{\eta_0}\left(1 - \frac{\sigma}{\sqrt{I}}\right) \\
I &= \frac{1}{2}\left(t_{ik} + t_{ki}\right) \\
\tau &= -p\delta_{ik} + t_{ik}
\end{aligned}
\qquad (1.2.2)
$$

where t_{ik} are the deviatoric stress tensor components, η_0 is the constant reciprocal mobility and σ the yield value of the material. Equations (1.2.2) are valid only when $\sqrt{I} \geqslant \sigma$. When $\sqrt{I} < \sigma$, the material behaves as an elastic solid. Examples of the fluids which have been stated to have approximately Bingham plastic behaviour are drilling muds, suspensions of chalk, grains, rocks, toothpastes, oils and

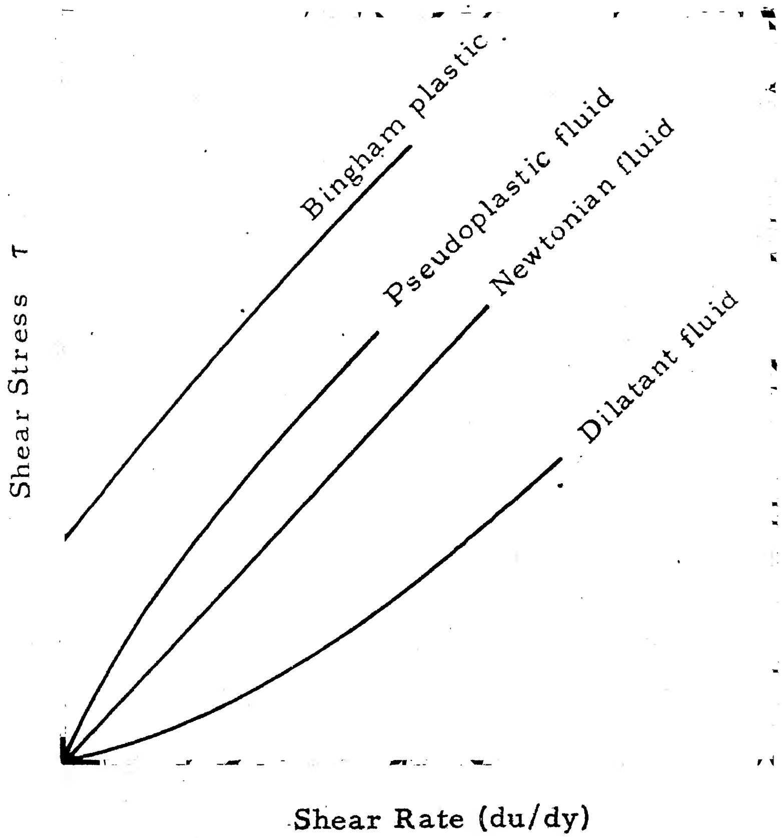

Figure 1.1 Arithmatic shear diagram for classical non-Newtonian fluids.

paints and sewage sludges etc.

Pseudoplastic fluids are second in importance only to Newtonian fluids. For one-dimensional flows they display the concave downward flow curve relationship (fig.1·1). For the general case the rheological equation for a pseudoplastic power-law fluid is given by the equation :

$$t_{ik} = K \left| \left(\sum_{l=1}^{3} \sum_{m=1}^{3} e_{lm} e_{ml} \right)^{\frac{n-1}{2}} \right| e_{ik} \qquad (1.2.3)$$

where K and n are called the consistancy and flow behaviour indices respectively and $n < 1$. A pseudoplastic fluid may be thought of as a fluid composed of solid and liquid particles, which does not dilate at rapid shear rates. The asymmetric particles of its solid content, at higher rates of shear align themselves along their major axes in the direction of the flow causing thereby a reduction of shear-stress as the shear-rate increases. By increasing the shear rate ultimately a situation may arise when no further alignment along the streamlines is possible. Then the fluid will cease to behave as pseudoplastic. It behaves thereafter as Newtonian fluid.

Equation (1.2.3) gives a dilatant power-law fluid when $n > 1$. Dilatancy occurs most frequently in suspensions of solids at high solid content. It signifies the quality of dilatation as was observed by Osborne Reynolds who first perceived the chemical justification why the stress in

such fluids increases fast with the shear-rate. These fluids, in their state of rest, have the least voidage between their solid particles, which remain filled with the liquid contained in the fluid. So when the fluids are sheared at low rates, the liquid lubricates the motion of one particle past the other and consequently, the stresses are small. But at higher rates of shear, the dense packing of the solid particles breaks and the material dilates slightly causing thereby greater voidage between the solid particles which because of not being fully filled by the liquid provides lesser lubrication to solid particles and thus greater stresses are experienced at the greater rates of shear.

VISCO-ELASTIC FLUIDS

A visco-elastic fluid possesses both the elastic and the viscous properties. When a visco-elastic fluid is flowing, certain amount of energy is stored up in the material as strain energy, while some energy is lost due to viscous dissipation. Oldroyd's fluid, Maxwell's fluid, Rivlin-Erickson fluids etc. are some of the visco-elastic fluids representing dilute suspensions of solid particles in viscous liquids or the emulsion and suspensions of one Newtonian liquid into another. Viscoelastic effects are observed in polymer solutions such as polymethyl-metherylate solutions and air in touline. All the visco-elastic fluids exibit the normal stress effects, and that these normal stress effects

do not necessarily depend solely on fluid elasticity, but this generally is the case, and the usual as well as the most outstanding and important examples of systems exhibiting normal stress effects are clearly elastic in nature.

The pioneering studies of Weissenberg demonstrated the 'climbing' of elastic fluids on to a shaft rotating within the fluid, in opposition to the centrifugal forces. This Weissenberg effect is also manifested by an axial tension in such a fluid; if it is sufficiently viscous to be ''cut'' (as in the case of molten polymer), the liquid that has climbed up the shaft may be observed to contract longitudinally.

TIME-DEPENDENT FLUIDS

Time-dependent non-Newtonian fluids exhibit a reversible change in the apparent viscosity of more complex fluids at constant temperature with the duration of shear. This definition excludes irreversible changes due to permanent alternations of particles or molecules within the fluid, and is limited to steady-state effects in contrast to the unsteady-state time effects associated with visco-elasticity. These fluids may further be subdivided into two classes :

(i) Thixotropic Fluids

(ii) Rheopectic Fluids

according as the shear stress decreases or increases with time

when the fluid is sheared at a constant rate. However, we shall not discuss the flow of this class of fluid.

In this thesis we have mainly confined ourselves to the study of only those types of non-Newtonian fluids viz.

(i) Ostwald-de-Waale power-law fluid

(ii) Special class of non-Newtonian fluid

(iii) Oldroyd's model of visco-elastic fluid

We shall now discuss the constitutive equations for these fluids :

1.3 CONSTITUTIVE EQUATIONS

(i) Ostwald-de-Waale Power-Law Fluid

A nearly endless variety of equations have been proposed for portrayal of purely viscous non-Newtonian flow behaviour, particularly for pseudo-plastic fluids. One of the models of sufficient interest is the Ostwald-de-Waale 'power-law' model that obeys the stress, strain-rate relationship characterised by the rheological equation :

$$t_{ik} = k \left(\sum_{l=1}^{3} \sum_{m=1}^{3} e_{lm} e_{ml} \right)^{\frac{n-1}{2}} e_{ik} \qquad (1.3.1)$$

where k and n are called the consistency and the flow behaviour indices respectively. If $n < 1$, the apparent viscosity decreases with increasing shear stress, and the fluid is called the pseudoplastic power-law fluid, whereas, if $n > 1$, the apparent viscosity increases with increasing shear stress and the fluid is called the dilatant power-law fluid.

In practical cases, the relation (1.3.1), though emperical, appears to represent a wide variety of non-Newtonian fluids better than most other proposed equations and certainly better than any other available two-constant equation. For this reason the power-law rheological model forms the major portion of this thesis. Obviously, the power-law model includes the special case when n is unity, and K is equal to the Newtonian coefficient of viscosity μ. Generally, we may say that n is an index to the degree of departure from Newtonian behaviour in the sense that farther n is removed from unity, above or below, the more pronounced becomes the non-Newtonian characteristic.

In process industries the dilatant fluids are much less common than pseudoplastic fluids but when the power-law is applicable, the mathematical treatment of both types may be more or less the same.

(ii) Special Type of Non-Newtonian Fluid:

It is well known that no real fluid can be said to satisfy one particular rheological equation completely for all the shear rates. There are some liquids which are nearly Newtonian under one particular set of conditions but may behave as power-law under another set of conditions. Ellis fluid and Eyring-Powell fluids are some examples of such fluids. In chapter VI of this thesis we have considered another example of such fluid which obeys the constitutive equation :

$$t_{ij} = \mu e_{ij} + \mu_a \left| \left(\sum_{l=1}^{3} \sum_{m=1}^{3} e_{lm} e_{ml} \right)^{\frac{n-1}{2}} \right| e_{ij} \qquad (1.3.2)$$

where e_{ij} are the strain-rate components, t_{ij} are the stress components; μ, μ_a and n are the constants depending on the flow behaviour of the fluid. This model at one extreme represents a Newtonian fluid when n equals unity and the coefficient of viscosity is $(\mu + \mu_a)$, while at the other extreme it represents a power-law fluid for $\mu = 0$. Further for the values of n less than unity and small rates of shear, we see that the second term in (1.3.2) dominates over the first whereas for large shear rates the first term dominates over the second. Similarly for the values of n greater than unity and small rates of shear the first term dominates whereas for large shear rates the second term dominates. Thus the model (1.3.2) can represent a wider range of behaviour than the power-law model itself.

(iii) Oldroyd's Model of Visco-elastic Fluid

Oldroyd, Strawbridge and Toms (1951) have shown that the behaviour of dilute (Ca 3 percent) solution of highly polymerised methyl-methacrylate in an organic liquid in a simple shearing motion can be approximately, at sufficiently small rates of shear, represented by the stress-strain law proposed by Jeffereys (1929)

$$t_{ij} + \lambda \dot{t}_{ij} = 2\mu \left(e_{ij} + \tau_0 \dot{e}_{ij} \right) \qquad (1.3.3)$$

where t_{ij} is the deviatoric stress-tensor, λ and τ_0 are the relaxation and retardation time parameter respectively of the liquid and μ is the coefficient of viscosity as in the case of Newtonian flow, e_{ij} is the rate of deformation tensor and the dot over it denotes the change and is the rate of stress and is equal to the expression.

$$\frac{\partial}{\partial t} t_{ij} + t_{ij,k}\, v^k - t_{ik}\, v_{j,k} - t_{kj}\, v_{i,k} + t_{ij}\, v_{k,k} \quad (1.3.4)$$

where

$$e_{ij} = \frac{1}{2}\left(v_{i,j} + v_{j,i} \right)$$

For incompressible liquids $v_{k,k} = 0$. In this generalisation, we cannot avoid non-linear equations of state if we remove all restrictions on the rate of strain. In (1.3.3) we have, therefore, considered only a particular case of this fluid where only the relaxation time parameter λ is taken into account.

Oldroyd (1950) has considered the problem of generalizing (1.3.3) subject to the considerations :

(i) The equations should be valid for all shear rates

(ii) The generalized equation should be in a form which does not change when coordinate system is changed, so that we do not need to select any particular coordinate system in special relationship with the flow pattern.

(iii) The translation as well as the rotation of the moving

element with the motion must be taken into account.

He has thus found that the equation of state for another isotropic incompressible Oldroyd liquid in Cartesian notations can be written as

$$p_{ik} + \lambda_1 \left[\frac{Dp_{ik}}{Dt} - p_{ij} e_{jk} - p_{jk} e_{ij} \right] + \mu_0 \, p_{ij} e_{jk} + \nu_1 \, p_{jl} e_{jl} \delta_{ik}$$

$$= 2\eta_0 \left[e_{ik} + \lambda_2 \left(\frac{De_{ik}}{Dt} - 2 e_{ij} e_{jk} \right) + \nu_2 \, e_{jl} e_{jl} \delta_{ik} \right] \qquad (1.3.6)$$

where λ_1 , λ_2 , μ_0 , ν_1 and ν_2 are 'time constants' and η_0 is the static or zero shear rate viscosity. The material derivatives are given by the expressions

$$\frac{Dp_{ij}}{Dt} \equiv \frac{\partial p_{ij}}{\partial t} + v^k \frac{\partial p_{ij}}{\partial x^k} - \omega_{ik} p_{kj} + \omega_{kj} p_{ik}$$

$$\frac{De_{ij}}{Dt} \equiv \frac{\partial e_{ij}}{\partial t} + v^k \frac{\partial e_{ij}}{\partial x^k} - \omega_{ik} e_{kj} + \omega_{kj} e_{ik}$$

where $\omega_{ij} = \frac{1}{2} \left(\frac{\partial v_i}{\partial x^j} - \frac{\partial v_j}{\partial x^i} \right)$ is the vorticity tensor,

δ_{ik} is the Kronecker tensor and $p_{ik} = t_{ik} + p\delta_{ik}$.

This form of the equation of state implies equal pressures normal to the flow direction in simple shear. For determining any flow, this has to be solved in conjuction with the usual continuity and equilibrium equations.

(iv) Rivlin-Ericksen Fluid

Rivlin (1948a, 1949b), Rivlin and Ericksen (1955) have shown that if we assume that in a visco-elastic fluid which is isotropic in its state of rest, the stress components are expressible as polynomials in the gradients of velocity, second acceleration ------$(n-1)^{th}$ acceleration at the point considered, the stress matrix $T = \| t_{ij} \|$ may be expressed as a matrix polynomial in n kinematic matrices $A_1, A_2, \ldots A_n$ whose coefficients are expressible as polynomials in traces of products formed out of these matrices. The Kinematic matrices are defined by equations:

$$A_r = \| A_{ij}^{(r)} \| \qquad (r = 1, 2, \cdots n) \qquad \textbf{(1.3.6)}$$

where

$$A^{(1)} = v_{i,j} + v_{j,i} = 2 e_{ij}$$

$$A_{ij}^{(r+1)} = \frac{\partial}{\partial t} A_{ij}^{(r)} + v^l A_{ij,l} + A_{mi}^{(r)} v_{m,j} + A_{mj}^{(r)} v_{m,i} \qquad \textbf{(1.3.7)}$$

$$(l, m = 1, 2, \cdots\cdots n)$$
$$(i, j = 1, 2, \cdots\cdots n)$$

Later on Rivlin (1956) considered certain simple types of steady state laminar flows viz rectilinear laminar flow, torsional flow between two-parallel plane discs, helical flow in the annular region between two coaxial cylinders, for

the viscoelastic fluids for $A_r = 0$ when $r > 2$ and

$$T = -pI + \alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_1^2 + \alpha_4 A_2^2$$

$$+\alpha_5 (A_1 A_2 + A_2 A_1) + \alpha_6 (A_1^2 A_2 + A_2^2 A_1)$$

$$+ \alpha_7 (A_1 A_2^2 + A_2 A_1^2) + \alpha_8 (A_1^2 A_2^2 + A_2^2 A_1^2) \qquad (1.3.8)$$

where I is the unit matrix and $\alpha_1, \alpha_2 \ldots \alpha_8$ are polynomials in the ten scalar invarients $tr\, A_1$, $tr\, A_1^2$, $tr\, A_1^3$, $tr\, A_2$, $tr\, A_2^2$, $tr\, A_2^3$, $tr\, A_1 A_2$, $tr\, A_1^2 A_2$, $tr\, A_1 A_2^2$ and $tr\, A_1^2 A_2^2$.

It is obvious, that Reiner-Rivlin fluid characterised by the constitutive equation

$$t_{ij} = -pI + \mu e_{ij} + \mu_c e_{im} e_{mj} \qquad (1.3.9)$$

where μ and μ_c are polynomials in invarients II and III defined by

$$II = \frac{1}{2} e_{ij} e_{ji}$$

$$III = \frac{1}{3} e_{ij} e_{jk} e_{ki} \qquad (1.3.10)$$

is only a particular case of (1.3.8) where $\alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_6 = \alpha_7 = \alpha_8 = 0$.

Kapur and Shashi Goel (1961) have obtained the conditions under which a rectilinear flow of the fluid can be maintained by a uniform pressure gradient.

1.4 BOUNDARY-LAYER FLOWS OF POWER-LAW FLUIDS

As early as 1926, the empirical relation (1.3.1) representing the rheological equation of a power-law fluid was used by Scott Blair (1938) but from an engineering point of view the recent studies by Salt (1949), Rankin (1955) and Weltman (1955) etc. may be of greater interest. Although Reiner (1946) has objected to this kind of empirical relation not derived from any physical concepts, even then in recent times this relation (1.3.1) has enabled us to explain various flow problems (See Dikshit (1966), Kapur (1963c), Kapur and Srivastava, P.N. (1963), Reiner (1960) and Proudman (1966) etc.) including the boundary layer flows, though in fact exact analysis of the flow situation for non-Newtonian fluids is possible in only very special situations. One of these refers to boundary layer flows of power-law fluids. The boundary layer theory for Newtonian fluids is well developed. In fact it tries to find out the asymptotic solutions of Navier-Stokes equations for large Reynolds number. When a liquid is flowing over a plate, the influence of the viscosity of the liquid is felt in a thin layer (the so-called the boundary layer) near the solid surface. Outside this layer, velocity gradients are small, and the theory of the potential flow is assumed to be valid. A number of assumptions are made to simplify these equations to find their solutions.

Boundary layer theories for power-law fluids have been developed by a number of workers (Acrivos, Peterson and Shah

(1960), Kapur (1963b) and Nanda (1962)). Bogue (1959) has used the boundary layer equations to extend Shiller's (1922) method for discussing the inlet length for a circular pipe for a Newtonian fluid. Kapur (1963c) has also integrated the boundary layer equations for the flow in a two-dimensional jet of an incompressible pseudoplastic power-law fluid. He has drawn the velocity profile for the particular case when the flow behaviour index n is equal to 1/3. The boundary layer thickness has been shown to vary as $(K)^{1/n+1}$ where K is the consistency index of the fluid. Kapur and Gupta (1963a) have integrated the boundary-layer equations to determine the entrance length and the boundary layer thickness as a function of the flow behaviour index n in two-dimensional flow of a power-law fluid in a straight channel. They (1963b) and the author have also found that the velocity profiles in the inlet length of a straight channel and circular pipe respectively can also be approximated by fourth degree velocity profiles. The laminar flow of power law fluids have also been discussed very widely in literature (Bizzel and Slattery (1962), Kapur (1963c), Shukla (1965), Srivastava (1966) and Williams and Bird (1962)).

The boundary layer equations for the two-dimensional flow of power-law fluid are given by

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = U(x) \frac{dU(x)}{dx} + \nu \left[\frac{\partial}{\partial y} \left| \left(\frac{\partial u}{\partial y} \right) \right|^{n-1} \frac{\partial u}{\partial y} \right] \qquad (1.4.1)$$

and $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

where u and v are the components of velocity along and perpendicular to the wall, $U(x)$ is the free stream velocity and $\nu = (2)^{\frac{n-1}{2}} \cdot \mu/\rho$, μ and ρ are the viscosity coefficient and the density respectively. These equations cannot in general be integrated exactly owing to the complexity of the term $(\partial u/\partial y)^n$ and of $U(x)$ being an arbitrary function. Integration of this equation by Karman-Pohlhausen method is not satisfactory for the entire range of n as has been shown by Bizzel and Slattery (1962). Abbot and Kline (1960), Manohar (1963) and Morgen (1952) have given the methods of reduction by one variable in the partial differential equations envolving more than one independent variables. On the basis of such methods, Schowalter (1960), Ishigawa (1963), Ingoff (1964), Okabe (1963) and many more have shown that the above two equations admit of ''Similar solutions''. Such solutions for the boundary layer equations of Newtonian fluids are well known Schlichting (1960). Schowalter was the first to give ''similar solutions'' for boundary layer equations of the pseudoplastic power-law fluid. Later on Tomita (1961) published a similar treatment. Though mathematically Schowalter's (1960) treatment was valid for both the values of n less than unity and also for greater than unity, yet he

confined himself to pseudoplastic $(n < 1)$ fluids. The reason was two-fold. Experimental studies have shown that power-law relation describes the behaviour of a number of real pseudoplastic fluids reasonably well in certain flow situations, the similar statement cannot be made for dilatant fluids $(n > 1)$. In particular serious difficulties arise when n approaches 2 as has been pointed out by Acrivos, Peterson and Shah (1960). At a later date Kapur and Tyagi (1964) have obtained the similar solutions of the boundary layer equations for two-dimensional flows of a general Reiner-Rivlin fluid. They have shown that these solutions exist if the free stream velocity is perpendicular to one-third power of the arc measured along the wall from the stagnation point. Nanda (1962) has shown that in the ease of Reiner-Rivlin fluid with constant coefficients of viscosity and cross-viscosity, the only possible similarity solutions for three-dimensional boundary layer equations in Cartesian coordinates are :

$$\text{Case (i)} \qquad \bar{U} = a\bar{x} + b\bar{z} \;, \bar{W} = n(a\bar{x} + b\bar{z})$$

$$\text{Case (ii)} \qquad U = m\bar{x} \;, W = n\bar{x} \qquad (1.4.2)$$

Acrivos, Peterson and Shah (1965) have developed an asymptotic method for solving the laminar boundary layer equations for power-law fluid under conditions where the flow external to the boundary layer has a general form, by the method of

''similar solutions'' . Brown and Stewartsen (1965) have also obtained ''similar solutions'' in which the vorticity decays algebraically towards the outer limits of the viscous layer and that the similarity solutions with algebraic decay can be the limit solutions of the full boundary layer equations with exponential decay.

In chapter II we have developed similar solutions of the boundary layer equations for power-law fluids when the velocity distribution of the potential flow is proportional to a power of the length of the arc measured along the wall from the stagnation point. A number of important particular cases including

(i) the boundary layer flows along a wedge

(ii) the boundary layer flows along a flat plate

(iii) the boundary layer flows in a convergent channel

and (iv) two-dimensional stagnation point flows

have been given. The treatment of this chapter was developed independently of the treatment of Schowalter (1964) and Tomita (1961). In fact, a later comparison between these results and those of Schowalter revealed a mathematical discrepency in his treatment (as reported by Skelland (1967)). This descrepency in the correct form for Schowalter's results has also been given in the last section of this chapter.

In chapter III while discussing the velocity profiles for the power-law fluid in the inlet length of a circular pipe, we have found that it can also be approximated by a cubic curve

and a fourth degree curve. In either case a free constant 'a' remains to be determined after satisfying the boundary conditions, and this constant 'a' we have determined by the principle of equal integrated kinematic energy and also by the principle of least squares method. These values of 'a' are fairly comparable to the approximation by Bogue (1959).

In chapter VI we have considered the rheological equation of state in which the stress is regarded as the linear combination of two terms; one indicating Newtonian behaviour and the other indicating the power-law behaviour. This represents Newtonian fluid at one extreme and the power-law at the other extreme, and it can also represent the entire spectrum of behaviour between these two extremes. One particular case of this has been used by Ng and Saibel (1962) and Kapur and Gupta (1965) to discuss the lubrication flow for the slider bearing. We have given the exact analytical solution for the flow of such a fluid between two plates and in the straight circular pipe. We have also discussed the series solutions for the fluids for which the Newtonian or the power-law terms can be regarded as a perturbation over the other more dominant term.

1.6 VISCO-ELASTIC FLUID FLOWS

Viscoelastic non-Newtonian flows have been discussed by Reiner (1946, 1960), Rivlin (1948a, 1948b), Oldroyd (1050, (1951), 1958) and others. Certain axially-symmetric problems

viz. flow through a circular tube have been considered by Rivlin and Green (1956), Bhatnagar and Rao (1957), Oberoi and Kapur (1960) and Schechter (1961). Flow in an annulus between two circular cylinders has been considered by Serrin (1959), Datta (1961), Fredrickson (1961), Narasimhan (1961), Rathna and Rajeshwari (1962) and Pipkin (1965). Langlois and Rivlin (1963) have considered the slow and steady state flow of a visco-elastic fluid characterised by Rivlin-Ericksen equation of state (1.3.8) through non-circular tubes. Sharma (1959b) has applied Karman-Pahlhausen method in solving the ordinary differential equation obtained in the study of rotation of a plane lamina in the visco-elastic fluid characterised by (1.3.4). Langlois (1963) has extended the Newtonian flow with negligible inertia considered by Proudman (1956) and Haberman (1962) between two spheres to the case of Rivlin-Ericksen fluid and have shown that the inclusion of the visco-elastic term in the equations of motion influences the flow between spheres in two distinct ways : Some of these terms modify the distribution of the rotatary flow about the axis, whereas the other of them contribute to the secondary flow in the meridinal planes. Kapur and Goel (1961a) have obtained the conditions under which the rectilinear flow of a visco-elastic Rivlin-Ericksen fluid can be maintained under a uniform pressure gradient under the same conditions as obtained by Rivlin and Green (1956) for the Reiner-Rivlin fluid. Rajeshwari and Rathna (1962) have followed the method of Srivastava, A.C.(1958)

in applying Karman-Pohlhausen method to obtain an approximate solution of the flow of a visco-elastic fluid (1.3.4) near a stagnation point. Datta (1964) has considered the flow of a particular visco-elastic fluid between two parallel porous plates with suction and injection. Sharma (1959a) has also considered flow of the same fluid near the stagnation point. Leslie and Tanner (1961) have investigated the slow flow of a visco-elastic liquid past a sphere. Mohan Rao (1962) has also discussed the flow of the same liquid between two rotating coaxial cones having the same vertex. Pipkin (1963) has shown that the rectilinear flow of non-linear visco-elastic fluid through straight tubes can only occure if the tube is of circular cross-section or if the rheological constants satisfy certain relations. Fredricksen (1961) and Tanner (1963) have discussed the helical flow of visco-elastic fluids in channels. Williams and Bird (1962a) have discussed the steady flow of Oldroyd's model of visco-elastic fluid in tubes, slits and narrow annulii. They (Williams and Bird (1962b)) have also studied the three constant Oldroyd's model for visco-elastic fluids. The flow of such fluids has also been discussed by Thomas and Walters (1963) in a curved pipe under a pressure gradient. Srivastava, P.N. (1963) has discussed the propagation of small disturbances in a semi-infinite visco-elastic liquid due to the slow angular motion of a disc in two cases viz. (i) when the angular velocity of the disc $\omega(t) = \omega_0 \delta(t)$ (Dirac Delta function) and (ii) $\omega(t) = \omega_0 \sin nt$.

The solution, as the relaxation time parameter λ tends to zero, has been shown to corresponds to the Newtonian case. Oberoi and Kapur (1960) gave some general considerations regarding axially-symmetric non-Newtonian flows and derived some new solutions for this problem. They have used the method generally used for discussing the axially symmetric viscous and axially symmetric hydromagnetic flows.

In chapter V we have considered the steady motion in an annulus and between two parallel plates of the viscoelastic fluid characterised by the rheological equation (1.3.4). The fluid is assumed to be injected at one of the boundaries at the same rate as it is withdrawn at the other, and one of the plates or the cylinders is moving with a constant velocity parallel to itself. Solutions have been obtained for small velocity of suction and injection for all relaxation time parameter and for small relaxation times for all velocities of suction and injection.

In chapter IV we have considered the axially-symmetric and two-dimensional stagnation point flows of the visco-elastic fluid characterised by Oldroyd's model (1.3.4) in which only the relaxation time parameter λ is taken into account. We have been able to give a sixth degree velocity profile to satisfy all the boundary conditions which were not fully satisfied by Sharma. It has been found that both the boundary layer thickness and the skin-friction decrease with the increase of the visco-elastic parameter - the results just reverse of

the corresponding results of Sharma but in agreement with the results of Rathna and Rajeshwari (1962) for another model. The reasons for this difference have also been examined. The two-dimensional stagnation point flow have also been considered.

1.6 SOME PROBLEMS ON REINER-RIVLIN AND RIVLIN-ERICKSON FLUIDS

In a paper we (1964) used Reiner-Rivlin fluid characterised by the rheological equation (1.3.9) with coefficients of viscosity and cross-viscosity as constants, as a lubricant in an externally pressurised bearing. There we found that both the flow rate of the lubricant and the load capacity decrease with the increase in the coefficient of cross-viscosity if the pressure in the recess is kept the same. If, however, this pressure is so adjusted as to ensure the same flow rate as for a Newtonian fluid, we found that the load capacity actually increases with the increase in the cross-viscosity coefficient. An additional effect of this presence of the cross-viscosity is that the surface of the liquid film exposed to the admospheric pressure p_0 given by $p = p_0$ are no longer circular cylinders though these are still the surfaces of revolution. Since these surfaces are concave outward, these are yet another explanation that these fluids exhibit the Weissenberg phenomena. In another paper we (1966) considered a number of possible flows for Reiner-Rivlin fluids when the coefficient of viscosity μ is constant and the

coefficient of cross-viscosity μ_c is a function of invariants (1.3.10). There we concluded that though it is satisfying to some extent to be able to extend the exact solutions for Newtonian fluids to the more general Reiner-Rivlin fluids (see Bhatnagar (1964)) such an extension is possible only for a very special class of those fluids for which μ and μ_c are constants. For two-dimensional flows, however, μ_c can be allowed to vary. For Poisseulle and Couette flows, however, it is possible to obtain solutions for the general Reiner-Rivlin fluids (see Reiner (1960)). It is, however, easy to see that in all the following cases :

(i) convergent and divergent flows between two non-parallel walls,

(ii) two-dimensional stagnation point flows,

(iii) axially-symmetric stagnation point flows

and (iv) flow about a rotating disc

extensions are not possible even if walls are porous.

We have not included the above two investigations in this thesis, as recently the very existence of such fluids has been denoted, since the dissipation function of such fluids can possibly be negative (see Kapur (1959)), which can never happen in physical world.

In another investigation we discussed the use of a slightly non-Newtonian liquid (Rivlin and Ericksen (1955) and Langlois (1963)) as lubricant in a slider bearing without any side leakage. The lower runner surface moves with a

constant velocity parallel to itself while the stationary shoe (finite in length) above it is inclined at a small angle α supports the load. The direction of the motion of the runner is towards the narrower face of the bearing. We found that the equipressure surfaces are no longer cylindrical in shape but still they are surfaces of revolution concave towards the direction of flow, and this again illustrates the Weissenberg phenomena exhibited by such fluids. In the expression for the load capacity of the bearing we saw that the contribution due to the elastic coefficient is always positive so that it increases the load capacity of the bearing, whereas for negative coefficient of cross-viscosity the contribution is still positive. The frictional resistence on the stationary shoe also decreases with increase in the elastic coefficient, and that the difference of frictional resistences on the two surfaces arise only due to the Newtonian viscous coefficient of the fluid.

1.7 FLOW IN CURVED CHANNELS

It can undoubtedly be said that the channel flow and heat transfer problem are frequently encountered and play a very important role in science and Technology. Mainly theoretical work has been done on three duct shape cases viz.

(i) straight channels

(ii) curved channels

(iii) convergent or divergent channels.

Comparatively, a very large number of theoretical studies have appeared on straight channels. The reason is that they are simplest to discuss mathematically. The curved channel case is very complicated from the point of view of theoretical handling. Consequently, a very small number of papers, dealing with curved channels are available.

In the seventh and the subsequent chapters of this thesis, we are interested only in the curved channels. Here a curved channel is a duct whose axis is not a straight line (as in the case of a straight channel) but a curved line, whose shape and cross-sectional area remain unchanged in the axial direction.

Curved channels are encountered in several branches of science. Specially, curved channel flow and heat-transfer problems are of great interest to biologists* and engineers. For instance, several types of curved ducts occure with in the human body (including the ducts through which blood flows) and in heat exchangers (in the heat exchange systems of various industrial equipments, curved channels act as heating or cooling coils). Construction of curved passages for the coolant in the heat exchange systems of various power producing mechanical units is now-a-days an important problem. Thus, curved channels case is of great technical importance. It is of particular interest from the theoretical point of

*One strong evidence of the interest of the medical scientists in flows through a curved pipe was obtained when we received a large number of requests, from medical institutes all over the world, for our paper on the subject published in Applied Scientific Research.

view also, since the field of curved channel flow and heat transfer problem presents many difficult problems.

The theoretical study of the subject of curved channel flow was started by Dean (1927). This pioneer paper deals with a laminar, steady, constant property and fully developed forced flow inside a curved pipe of simply connected circular cross-section whose axis forms a circular arc. The necessary assumption made in this paper is that the curvature of the channel is so small that squares and higher powers of the ratio of the radius of the cross-section to that of the radius of curvature of the arc formed by the axis can be neglected. The theory thus gives us an approximation of the first order. It is in good qualitative agreement with experiment of Eustice (1911). However, the first experimental study of the subject was made by Grindley and Gibson (1908). The theory of Dean (1927) does not admit the dependence of the flow rate and of the resistance coefficient on pipe curvature. The next paper of Dean (1928) shows these dependences. The theory of this paper is based on a closer approximation than that of his earlier paper of 1927. This extension, so far as it has been carried, represents a fourth approximation to the problem. Dean (1927, 1928) point out that the representative number of the problem(called Dean number) is aR^2/ℓ . Obviously Dean's studies are valid only for small values of this number. Dean (1928) showed that the flow rate and the resistence coefficient are

functions of Dean number alone. This was well confirmed by experimental work of White (1929). As a matter of fact, the aim of the experimental study of White was to test the Dean's studies. White (1929) was in complete agreement with Dean (1927,28) and also gave values of Dean number for which Dean's work is valid. For a large Dean number, we have the first study of Adler (1934). This contains both theoretical and experimental investigations. The theory is based on the supposition of the existence of a thin boundary layer along the wall. Adler's resistance coefficient formula asymptotically approaches the observed results in very large Dean number range. Works of White and Adler show that the variation of resistence coefficient with pipe curvature is negligible upto certain small values of Dean number. Experiments of Haves (1930) and Squire (1954) show that in large Dean number case, various forces are important only in a thin boundary layer near the wall (see Barua (1963)). Studies of Haves and Squire motivated Barua (1963), who assumed that the fluid motion in the vicinity of the wall occur within a thin boundary layer adhering to the wall and the fluid motion outside this layer is confined to the plane parallel to the plane of symmetry of the channel. The resistence coefficient thus obtained agrees with the observations of White and Adler. In the analytical works of Adler and Barua, some of the boundary conditions are not satisfied at the edge of the boundary, the velocity profile of the inside boundary

layer region is not joined smoothly to that of the outside boundary layer region and the circumferential velocity does not vanish at $\phi = -\pi/2$ (or $3\pi/2$). Adler and Barua do not give complete or approximate momentum balance analysis for velocity profile and introduce discontinuity of tangential velocity at the edge of the boundary layer and, therefore, they get a finite tangential velocity in the boundary layer at $\phi = -\pi/2$. These defects in the analysis of Adler and Barua have been removed in a recent work of Mori and Nakayama (1965). Also, this paper deals with a heat transfer problem and provides experimental observation for both flow and heat transfer problems. It should be noted that studies of Dean, are based on perturbation technique (i.e. technique of successive approximations) whereas those of Adler (1934), Barua (1963) and Mori and Nakayama (1965) make use of the technique of writing momentum and the energy integral boundary layer equations and the Kármán-Pohlhausen technique. The theoretical studies referred to in the foregoing, deal with laminar flow case only. Most of the studies dealing with turbulent flow case are experimental viz. Jeschke (1925), Nippert (1929), Richter (1930), Detra (1953), White (1932), Cummings (1955) etc. Theoretical analysis of Ito (1959) and Mori and Nakayama (1967a) are based on the boundary layer concept, and the Karman-Pohlhausen technique. The two studies give some experimental results also.

Following Dean's technique, some extensions of Dean (1927) are also available, e.g. we have Jones (1960) for

Reiner-Rivlin fluid with constant coefficients of viscosity and cross-viscosity, Clegg and Powers (1963) for Bingham plastic fluids, Thomas and Walters (1963) for an elastico-viscous liquid, Thomas and Walters (1964) for an elastico-viscous liquid and elliptic cross-section, Pathak (1966) for Reiner-Rivlin fluid of constant coefficients of viscosities and curved annulus and Chaudhary (1964) for the motion of a viscous liquid in a curved pipe of the cross-section of the form $r = f(\theta)$. Our own studies, which are also based on Dean's technique are given in chapter VII.

In the seventh chapter we have carried out theoretical calculations for the steady, laminar, incompressible flow of an ordinary viscous Newtonian fluid in a curved annulus. Corresponding to the stream function ψ, pressure gradient p and the velocity ω for the case of a straight annulus, we have found (following Dean (1927)), ψ', p' and ω' respectively for the curved annulus on the supposition of secondary flows developed in the flow region due to the curvature which has been assumed to be small. At first the flow parameter are evaluated for the general case and then a particular example is solved numerically. We have carried out our calculations on IBM 7044 Computer to find out the velocity field inside the annular space, and the corresponding superposition on the velocity field due to the curvature of the pipe. We find that on the concave side of the horizontal diameter the superposed velocity slows down the fluid particles

while on the convex side of the pipe it gives a little acceleration to them. The graphical representation of the flow line in the plane of symmetry and the projection of the streamlines on a normal cross-section are also given. However, we have confined to the case of large radius of curvature of the pipe.

1.8 CONVECTIVE HEAT-TRANSFER IN CURVED CHANNELS

The curved channel heat transfer problem is of great technical importance, since the coils of tubes are largely used, for example, in connection with heating and refrigeration in order to transfer heat from one fluid to another. Under such conditions, the local heat transfer rate distribution on the curved channel boundary will in general be quite different from that on the corresponding straight channel boundary. Thus the heat transfer problems associated with curved channels are of importance for heat exchanger technology. Curved channel heat transfer problems are also found in heat engines and such various very industrial equipment as spiral tube heat exchangers. The removal of heat by means of a coolant flowing through curved passages in gas turbines and chemical and nuclear reactors may be said to be an important problem in the field of heat transfer studies.

Few studies on the problem are available in the literature. Of these, mostly they are experimental (as a matter of fact the very first study is an experimental one)

i.e. Jeschke (1926), Merkel (1927), White (1932), Hawes (1932), Ede (1961), Seban and Mc Laughlin (1963), Rogers and Mayhew (1964) etc. The first theoretical study in the field of curved channel heat transfer is that of Mori and Nakayama (1965). After this, there are two more theoretical studies in our knowledge, namely Mori and Nakayama (1967a) and Mori and Nakayama (1967b). The first two of these studies of Mori and Nakayama are experimental also. Each of these three papers deals with Dean's curved pipe geometry. Mori and Nakayama (1965) give theoretical and experimental investigations of heat transfer associated with a laminar, steady, constant property fluid and fully developed (both hydrodynamically and thermally) forced convection under the uniform wall heat flux condition. The purpose of Mori and Nakayama (1967a) is to extend their 1965 paper to include the case of turbulent forced convection. The purpose of Mori and Nakayama (1967b) is to give theoretical solution of the problems for hydrodynamically fully developed but thermally developing forced convection under the conditions of constant wall temperature. In each of these studies Mori and Nakayama, Dean number has been assumed to be very large and boundary layer concept has been used both for velocity and temperature fields.

In these theoretical studies, heat generation as a result of dissipation of mechanical energy due to viscosity property has not been considered for curved pipe case.

An analysis of heat generation due to dissipation for curved pipe is given in the eighth chapter of this thesis. In the last chapter of this thesis we shall be discussing that last of our studies i.e. the analysis of convective heat transfer due to the viscous dissipation in curved annulus streamlines in the field of curved channel flow and heat transfer which were proposed to be given in the second part of this thesis.

In the eighth chapter we have considered the convective heat transfer flow of an incompressible Newtonian liquid under a constant pressure gradient through a circular pipe whose axis forms a circle of large radius ℓ, much larger than the radius of the cross-section of the pipe. The temperature of the wall is kept constant both with respect to time as well as with respect to direction, and both the velocity and the temperature profiles are fully developed. We have adopted the method of imposing a perturation temperature over the known results for corresponding heat-transfer problem for a straight circular pipe. We have been able to find the temperature profile at all points of the cross-section of the pipe. Even the temperature inside the pipe attains a higher value than that at the wall, and this point goes on shifting towards the wall as we go on increasing the value of R the Reynold's number and that at no point on the concave side of the horizontal diameter, the temperature is above the wall temperature. Similar is the effect of the Prandtl number P_r, but not of equal percentage than the

Reynold's number R. This has been shown to be due to the curvature of the pipe as well. We have also calculated the skin friction on both the convex side and the concave side of the pipe and found that the higher temperature difference in the liquid on both sides of the vertical diameter is more due to the value of the Prandtl number than due to the rise in the skin friction, because the skin friction is higher on the convex side of the pipe than on the concave side.

The second study on the convective heat-transfer in curved channels contained in the second part of this thesis is that of the convective heat-transfer in a curved annulus. It was just for the sake of simplicity we have included (in chapter VIII) the heat-transfer in a curved pipe in between the two types of studies on curved annulus the later one is discussed in the ninth chapter.

Here the curved annulus is of the same geometry as that considered in the seventh chapter. We have considered the fully developed (both hydrodynamically and thermally) laminar forced flow. The method of approach to the problem is the same as adopted in the case of heat-transfer in a curved pipe in the preceeding chapter. The walls of the annulus for the sake of simplicity are kept at the same temperatures. We have evaluated the temperature profile throughout the annular space, and also the local heat transfer rate at both the inner and the outer walls of the annulus.

CHAPTER II

SIMILAR SOLUTIONS OF THE BOUNDARY LAYER EQUATIONS FOR POWER-LAW FLUIDS

2.1 INTRODUCTION

In this chapter, we discuss the theory for "similar solutions" of the boundary layer equations for power-law fluids on the same lines as is usually done (cf. Schlichting (1960)) for Newtonian fluids. In this manner we obtain a generalisation of the Falker-Skan equation. Important particular cases like the

(i) boundary layer flows along a wedge

(ii) boundary layer flows along a flat plate

(iii) boundary layer flows in a convergent channel

and (iv) two-dimensional stagnation point flows

have briefly been discussed.

2.2 BASIC EQUATIONS

The boundary layer equations for the two-dimensional flow of a power-law fluid are, as obtained by Kapur (1963c)

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U\frac{dU}{dx}+\nu\frac{\partial}{\partial y}\left[\left|\frac{\partial u}{\partial y}\right|^{n-1}\cdot\frac{\partial u}{\partial y}\right] \tag{2.2.1}$$

and

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{2.2.2}$$

where u and v are the components of velocity along and perpendicular to the wall. $U(x)$ is the free stream velocity, ρ is the density of the fluid and $\nu = 2^{\frac{n-1}{2}}.\mu/\rho$.

The equation of continuity is integrated by introducing the stream function $\psi(x,y)$ which is such that

$$u=\frac{\partial\psi}{\partial y}\quad;\quad v=-\frac{\partial\psi}{\partial x} \tag{2.2.3}$$

The boundary layer equations then reduce to

$$\frac{\partial\psi}{\partial y}\cdot\frac{\partial^2\psi}{\partial x\partial y}-\frac{\partial\psi}{\partial x}\cdot\frac{\partial^2\psi}{\partial y^2}=U(x)\frac{dU(x)}{dx}+\nu\frac{\partial}{\partial y}\left[\left|\frac{\partial^2\psi}{\partial y^2}\right|^{n-1}\cdot\frac{\partial^2\psi}{\partial y^2}\right] \tag{2.2.4}$$

We introduce new variables and functions :

$$\xi=\frac{x}{L}\quad;\quad\eta=\frac{y.R_1^{\frac{1}{n+1}}}{L\,g(x)} \tag{2.2.5}$$

$$f(\xi,\eta) = \frac{\psi(x,y)\ R_1^{\frac{1}{n+1}}}{L \cdot U(x) \cdot g(x)} \tag{2.2.6}$$

where R_1 is the modified Reynold's number defined by

$$R_1 = \frac{L^n}{\nu\ U_\infty^{n-2}} \tag{2.2.7}$$

L and U_∞ are the reference length and velocity respectively, and $g(x)$ is a suitable scaling function to be chosen later. In terms of these variables, we get

$$u = \frac{\partial \psi}{\partial y} = U(x) \frac{\partial f}{\partial \eta} \tag{2.2.8}$$

$$v = -\frac{\partial \psi}{\partial x} = -\left[L \cdot f \frac{d}{dx}(Ug) + Ug\left(\frac{\partial f}{\partial \xi} - \frac{g'}{g} L \eta \frac{\partial f}{\partial \eta}\right)\right] / R^{\frac{1}{n+1}} \tag{2.2.9}$$

$$\frac{\partial u}{\partial x} = U'(x)\frac{\partial f}{\partial \eta} + \frac{U(x)}{L}\left[\frac{\partial^2 f}{\partial \xi \partial \eta} - L\eta \frac{\partial^2 f}{\partial \eta^2} \cdot \frac{g'(x)}{g(x)}\right] \tag{2.2.10}$$

$$\frac{\partial u}{\partial y} = U(x) \cdot \frac{\partial^2 f}{\partial \eta^2} \cdot \frac{\eta}{y} \tag{2.2.11}$$

Substituting in (2.2.1) and remembering that for ''similar solutions'' $f(x, y)$ should be independent of ξ, we get after consideration simplifications, the basic equation as :

$$|f''|^{n-1} \cdot f''' + \alpha f f'' + \beta (1 - f'^2) = 0 \qquad (2.2.12)$$

where

$$\alpha = \frac{L g^n_{(x)} \, U_\infty^{n-2}}{n \, U^{n-1}_{(x)}} \frac{d}{dx}\left(U_{(x)} \cdot g_{(x)}\right) \qquad (2.2.13)$$

and

$$\beta = \frac{L \cdot g^{n+1}_{(x)} \cdot U_\infty^{n-2}}{n \, U^{n-1}_{(x)}} \cdot U'_{(x)} \qquad (2.2.14)$$

We choose $U(x)$ and $g(x)$ so that α and β are constants. For $n = 1$, the equation (2.2.12) reduces to the well known Falker-skan equations

$$f''' + \alpha f f'' + \beta (1 - f'^2) = 0 \qquad (2.2.15)$$

with

$$\alpha = \frac{L g_{(x)}}{U_\infty} \frac{d}{dx}\left(U_{(x)} \cdot g_{(x)}\right)$$

$$\beta = \frac{L g^2_{(x)} \, U'_{(x)}}{U_\infty} \qquad (2.2.16)$$

In general the velocity component u will increase from its zero value at the wall to the value $U_{(x)}$ at the edge of the boundary layer and thus in this case $\partial u / \partial y$ would be non-negative. From (2.2.6) and (2.2.11) it would then appear that if $g(x)$ can be chosen to be a non-negative

function, we can take f'' to be non-negative. Thus we can make the assumption that f'' is non-negative function and integrate

$$f''^{\,n-1} \cdot f''' + \alpha f f'' + \beta (1 - f'^2) = 0 \qquad (2.2.17)$$

subject to the boundary conditions

$$\eta = 0 \quad \text{for} \quad f = 0 \quad \text{and} \quad f' = 0$$

$$\eta = \infty \quad \text{for} \quad f' = 1 \qquad (2.2.18)$$

The assumption made can then be tested against the solution so obtained.

2.3 SOLUTION FOR $U(x)$ AND $g(x)$

From (2.2.13) and (2.2.14) we have

$$\frac{d}{dx}\left[g(x)^{n+1} \cdot U(x)^{2-n} \right] = \frac{n}{L U_\infty^{n-2}} \left[(n+1)\alpha - (2n-1)\beta \right] \qquad (2.3.1)$$

Integrating the above equation for

$$(n+1)\alpha - (2n-1)\beta \neq 0$$

we get

$$g(x)^{n+1} \cdot U(x)^{2-n} = \frac{n}{L U_\infty^{n-2}} \left[(n+1)\alpha - (2n-1)\beta \right] x \qquad (2.3.2)$$

Also from (2.2.13) and (2.2.14) we get

$$\frac{U'(x)}{U(x)}(\alpha-\beta) = \beta\frac{g'(x)}{g(x)} \tag{2.3.3}$$

which gives on integration

$$\left(\frac{U(x)}{U_\infty}\right)^{\alpha-\beta} = K\, g^{\beta}(x) \tag{2.3.4}$$

where K is a dimensionless constant.

Solving for $U(x)$ and $g(x)$ we get

$$\left(\frac{U(x)}{U_\infty}\right)^{\frac{\alpha-\beta}{\beta}+\frac{2-n}{n+1}} = K^{\frac{1}{\beta}}\left[\left\{(n+1)\alpha-(2n-1)\beta\right\}\frac{nx}{L}\right]^{\frac{1}{n+1}} \tag{2.3.5}$$

and

$$\left[g(x)\right]^{1+\frac{\beta(2-n)}{(\alpha-\beta)(n+1)}} = K^{-\frac{2-n}{(n+1)(\alpha-\beta)}}\left[\left\{(n+1)\alpha-(2n-1)\beta\right\}\frac{nx}{L}\right]^{\frac{1}{n+1}} \tag{2.3.6}$$

From (2.2.13) and (2.2.14) it is seen that the result is independent of any common factor of α and β, as it can be included in $g(x)$. Therefore as long as $\alpha \neq 0$, we can put $\alpha = 1$ without loss of generality. Also introducing a new parameter m defined by

$$m = \frac{\beta}{(n+1)\alpha-(2n-1)\beta} \quad \text{or} \quad \beta = \frac{m(n+1)}{1+m(2n-1)} \tag{2.3.7}$$

we get

$$\left(\frac{U(x)}{U_\infty}\right)^{1+m(2n-1)} = (k) \cdot \left[\frac{n(n+1)}{1+m(2n-1)} \cdot \frac{x}{L}\right]^m \qquad (2.3.8)$$

or simply $\quad U(x) = c \cdot x^m$

where c is a constant of proportionality.

and $$g(x) = \left[\frac{n(n+1)}{1+m(2n-1)} \cdot \frac{x}{L} \cdot \left(\frac{U_\infty}{U(x)}\right)^{2-n}\right]^{\frac{1}{n+1}} \qquad (2.3.9)$$

or simply $\quad g(x) = C_1 (x)^{\frac{1-2m+mn}{n+1}}$

where C_1 is a constant of proportionality.

Also from (2.2.5) and (2.3.9)

$$\eta = \left[\frac{1+m(2n-1)}{n(n+1)} \frac{U_\infty^{2-n}}{x^{\nu}}\right]^{\frac{1}{n+1}} \qquad (2.3.10)$$

The case $(n+1)\alpha - (2n-1)\beta = 0$ left earlier leads to the

$$g^{n+1}(x) \cdot U^{2-n}(x) = \text{constant} = C \qquad (2.3.11)$$

or

$$(n+1) \cdot g^n(x) \cdot g'(x) = C(n-2)\, U^{n-3}(x)\, U'(x)$$

and $$(\alpha-\beta) = C \cdot L \cdot U_\infty^{n-2} \frac{(n-2)}{n(n+1)} \frac{U'(x)}{U(x)} \qquad (2.3.12)$$

Integration of (2.3.12) gives

$$U(x) = c_2 \, e^{\left(\frac{1-2m+mn}{n+1}\right)x} \tag{2.3.13}$$

and also

$$g(x) = c_3 \, e^{\frac{(n-2)(1-2m+mn)}{(n+1)^2} \cdot x} \tag{2.3.14}$$

where c_2 and c_3 are constants of proportionality.

2.4 PARTICULAR CASES

Case I : Flow Past a Wedge

If the angle of the wedge is $\pi\theta$, the potential flow is given by

$$U(x) = c \, (x)^{\frac{\theta}{2-\theta}} = c \, x^m \tag{2.4.1}$$

From (2.3.7) and (2.3.11) we get

$$m = \frac{\theta}{2-\theta} = \frac{\beta}{[(n+1)\alpha-(2n-1)\beta]} \tag{2.4.2}$$

or

$$\beta = \frac{(n+1)\theta}{2\theta(n-1)+2} \quad \text{or} \quad \theta = \frac{2\beta}{[(n+1)\alpha-(2n-1)\beta]} \tag{2.4.3}$$

where α is taken to be 1 .

Also from (2.3.10), (2.3.9), (2.2.6), (2.2.8), (2.2.9) and (2.2.16) after considerable simplifications we have

$$\eta = y\left[\frac{1+m(2n-1)}{n(n+1)} \cdot \frac{c^{2-n}}{\nu}\right]^{\frac{1}{n+1}} \cdot (x)^{\frac{m(2n-1)-1}{n+1}} \tag{2.4.4}$$

$$\psi = \left[\frac{n(n+1)}{1+m(2n-1)} \cdot \nu \cdot c^{2n-1}\right]^{\frac{1}{n+1}} \cdot (x)^{\frac{1+m(2n-1)}{1+n}} \cdot f(\eta) \tag{2.4.5}$$

$$u = c \cdot x^{m} \cdot f(\eta) \tag{2.4.6}$$

$$v = -\left[\frac{n(n+1)}{1+m(2n-1)} \cdot \nu \cdot c^{2n-1}\right] \cdot (x)^{\frac{m(2n-1)-n}{n+1}} \cdot x$$

$$x\left[\eta \cdot \frac{\partial f}{\partial \eta} \cdot \frac{m(2n-1)-1}{n+1} + \frac{m(2n-1)+1}{n+1} f\right] \tag{2.4.7}$$

$$f''^{\,n-1} \cdot f''' + ff'' + \frac{(n+1)\theta}{2+2\theta(n-1)} \cdot (1-f'^2) = 0 \tag{2.4.8}$$

Integration of (2.4.8) subject to (2.2.18) would give $f(\eta)$ and $\frac{\partial f}{\partial \eta}$ and then (2.4.6) and (2.4.7) would give u and v. We can discuss both the accelerating flow $(m > 0)$ and the decelerating flow $(m < 0)$.

Case II : Two-dimensional Stagnation Point Flow

In the above analysis, we put $m = \beta = \theta = 1$, then we get

$$\eta = y \left[\frac{2}{n+1} \cdot \frac{c^{2-n}}{n\nu} \right] (x)^{\frac{1-n}{1+n}} \qquad (2.4.9)$$

$$\psi = \left[\frac{n+1}{2} \cdot \nu \cdot c^{2n-1} \right]^{\frac{1}{n+1}} \cdot (x)^{\frac{2n}{1+n}} \qquad (2.4.10)$$

$$u = c . x . f(\eta) \qquad (2.4.11)$$

$$v = -\left(\frac{n+1}{2} . \nu . c^{2n-1} \right)^{\frac{1}{n+1}} \left[\eta \frac{\partial f}{\partial \eta} \cdot \frac{1-n}{1+n} + f \frac{2n}{1+n} \right] (x)^{\frac{n-1}{n+1}} \qquad (2.4.12)$$

$$f''^{\,n-1} \cdot f''' + ff'' + \frac{n+1}{2n-1} \left(1 - f'^2 \right) = 0 \qquad (2.4.13)$$

We thus get the stagnation point boundary layer though unlike the Newtonian case, this may not satisfy the complete equations of motion.

Case III : Flow Past a Flat Plate

In this case we put

$$m = \beta = \theta = 0 \quad \text{then}$$

$$U(x) = c.(x)^0 = c = U_\infty \quad \text{(say)}$$

This gives us the case of a flat plate with zero incidence, and we get

$$\eta = y\left[-\frac{U_\infty^{n-2}}{\nu . n . (n+1)}\right]^{\frac{1}{n+1}} . (x)^{-\frac{1}{n+1}} \tag{2.4.14}$$

$$\psi = \left[n(n+1).\nu . c^{2n-1}\right]^{\frac{1}{n+1}} . (x)^{\frac{1}{n+1}} \tag{2.4.15}$$

$$u = U_\infty . f(\eta) \tag{2.4.16}$$

$$\vartheta = -\frac{1}{n+1}\left[n(n+1).\nu . c^{2n-1}\right]^{\frac{1}{n+1}} . \left[f - \eta\frac{\partial f}{\partial \eta}\right](x)^{-\frac{n}{n+1}} \tag{2.4.17}$$

$$f''^{\,n-2} . f''' + f = 0 \tag{2.4.18}$$

Case IV : Flow in a Convergent Channel

This case corresponds to $\alpha = 0$, $\beta = 1$. Then from (2.2.13) we get $U(x) . g(x)$ = constant, and from (2.2.14)

$$U(x) \propto (x)^{-\frac{1}{2n-1}} , \quad g(x) \propto (x)^{\frac{1}{2n-1}} \tag{2.4.19}$$

The basic differential equation becomes

$$f''^{\,n-1} . f''' + (1 - f'^2) = 0 \tag{2.4.20}$$

Multiplying (2.4.20) by f'' , integrating and using the condition that as $\eta \to \infty$, $f' = 1$, $f'' = 0$; we get

$$f''^{\,n+1} = \frac{n+1}{3}\,(1-f'^2)(2+f') \qquad (2.4.21)$$

Integrating this again

$$\eta = \left(\frac{3}{n+1}\right)^{\frac{1}{n+1}} \int_0^{f'} \frac{df'}{\left[(1-f'^2)(2+f')\right]^{\frac{1}{n+1}}} \qquad (2.4.22)$$

Since at $\eta = 0$, $f' = 0$. Further if $n \leqslant 1$ as $f' \to 1$, η would tend to infinity. If $n > 1$ when $f' \to 1$, η would be finite and after that f' would remain 1 .

We may also note that since

$$\frac{\partial u}{\partial y} = \frac{\partial^2 f}{\partial \eta^2} \cdot \frac{R_1^{\frac{1}{n+1}}}{L\,g(x)} \qquad (2.4.23)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^3 f}{\partial \eta^3} \cdot \frac{R_1^{\frac{2}{n+1}}}{L^2\, g^2(x)} \cdot \frac{1}{U(x)} \qquad (2.4.24)$$

for $n \leqslant 1$, we shall require both f'' and f''' tend to zero in such a way that $f''^{\,n-1} \cdot f'''$ also tends to zero as $\eta \to \infty$.

If $n > 1$, we need not insist on f''' tending to zero.

From (2.2.22)

$$\eta_1 = \left(\frac{3^{n-1}}{n+1}\right)^{\frac{1}{n+1}} \int_{2/3}^{\frac{f'+2}{3}} Z^{-\frac{1}{n+1}} \,(1-Z)^{-\frac{2}{n+1}}\, dZ \qquad (2.4.26)$$

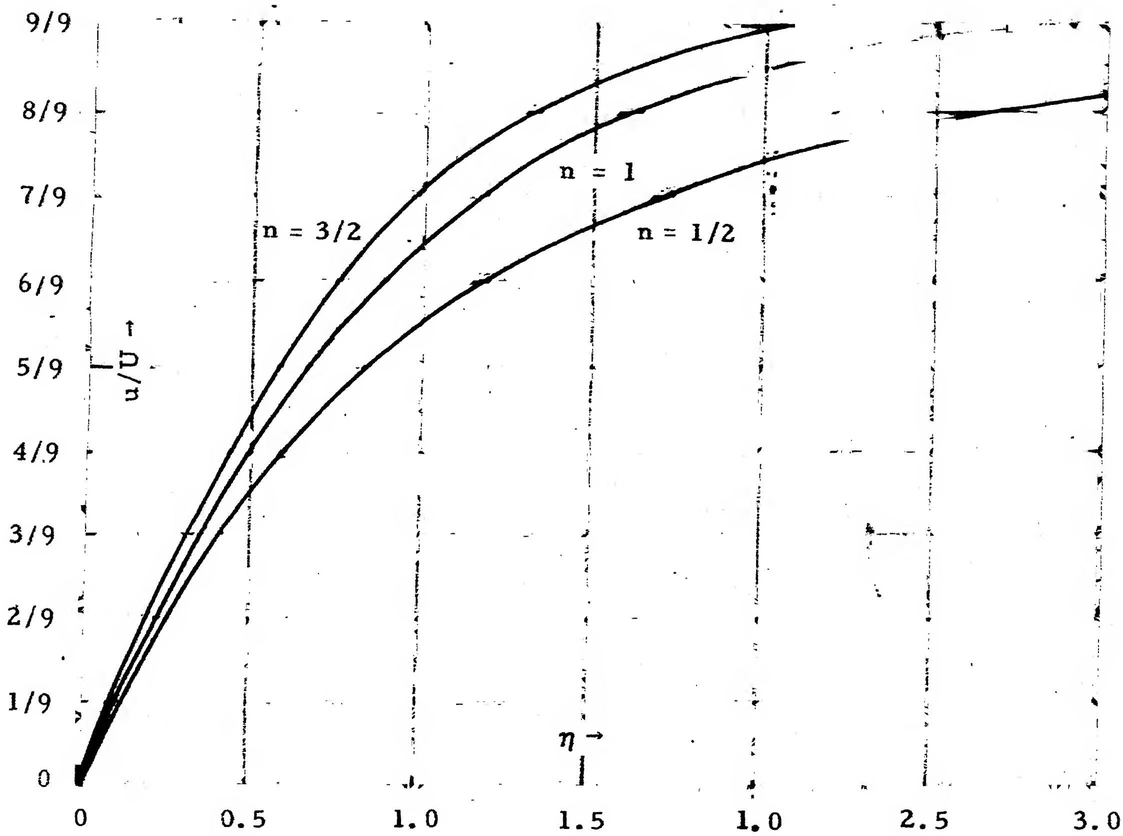

Figure 2. 1 Value of u/U as a function of η for pseudoplastic fluid (n = 1/2), a Newtonian fluid (n = 1) and a dilatant fluid (n = 3/2).

or

$$\eta = \left(\frac{3^{n-1}}{n+1} \right)^{\frac{1}{n+1}} \left[B_{\frac{f'+2}{3}} \left(\frac{n}{n+1} , \frac{n-1}{n+1} \right) - B_{\frac{2}{3}} \left(\frac{n}{n+1} , \frac{n-1}{n+1} \right) \right] \qquad (2.4.26)$$

where the incomplete $\beta -$ functions are defined by

$$B_x \ (p, q) \quad = \int_0^x x^{p-1} . (1-x)^{q-1} \ dx \qquad (2.4.27)$$

(2.4.25) or (2.4.26) enable us to plot u/U as a function of η . If $n > 1$, the tables of incomplete beta functions (see Pearson (1932) and (1934)) can be used. If $0 \leq n \leq 1$, numerical integration has to be used.

The graph (2.1) gives u/U as a function of η for pseudoplastic fluid $(n = \frac{1}{2})$, a Newtonian fluid $(n = 1)$ and a dilatant fluid $(n = \frac{3}{2})$.

2.5 SCHOWALTER'S DISCUSSION OF SIMILARITY SOLUTIONS

Schowalter (1960) has independently attacked the problem of similarity solutions in three-dimensional case of the flow past a flat plate where potential velocity vector is not perpendicular to the leading edge, and this is a much more restrictive result than is obtained for Newtonian fluids in three-dimensional flow.

The object of the present section is to point out a discrepency in the mathematical development of Schowalters (1960) and to give necessary corrections.

We use the notations of Schowalter's paper. Equations (14), (15) and (16) of the text are

$$\frac{g^{n+1}}{(U^\circ)^{n-1}}\frac{\partial U^\circ}{\partial x^\circ}\cdot\left[F'^2-FF''-1\right]+\frac{g^{n+1}}{U^\circ}\cdot W^\circ\frac{\partial U^\circ}{\partial z^\circ}\left[FG'-1\right]$$

$$-\frac{g^{n+1}}{(U^\circ)^{n-1}}\frac{\partial W^\circ}{\partial z^\circ}GF''-\frac{g^{n}}{(U^\circ)^{n-2}}\frac{\partial g}{\partial x^\circ}FF''-\frac{g^{n}W^\circ}{(U^\circ)^{n-1}}\frac{\partial g}{\partial x^\circ}F''G$$

$$=(R)^{N(n+1)-1}\cdot\frac{\partial}{\partial\eta}\left\{\left[F''^2+\left(\frac{W}{U}G''\right)^2\right]^{\frac{n-1}{2}}F''\right\}+\frac{KL}{U_\infty^2}\frac{g^{n+1}}{(U^\circ)^n}f_{2x};$$

$$\frac{1}{(U^\circ)^{n-1}}\cdot\frac{\partial U^\circ}{\partial x^\circ}\cdot\left[F'^2-FF''-1\right]+W^\circ\frac{\partial\ln U^\circ}{\partial z^\circ}\left[FG'-1\right]$$

$$-\frac{1}{(U^\circ)^{n-1}}\cdot\frac{\partial W^\circ}{\partial z^\circ}GF''-\frac{1}{(U^\circ)^{n-2}}\frac{\partial\ln g}{\partial x^\circ}FF''-\frac{W^\circ}{(U^\circ)^{n-1}}\frac{\partial\ln g}{\partial z^\circ}F''G$$

$$=\frac{1}{g^{n+1}}\frac{\partial}{\partial\eta}\left\{\left[F''^2+\left(\frac{W}{U}G''\right)^2\right]^{\frac{n-1}{2}}\cdot F''\right\}+\frac{KL}{U_\infty^2(U^\circ)^n}\cdot f_{2x};$$

$$\frac{1}{(W^\circ)^{n-1}}\frac{\partial W^\circ}{\partial z^\circ}\left[G'^2-GG''-1\right]+U^\circ\frac{\partial\ln W^\circ}{\partial x^\circ}\left[F'G'-1\right]$$

$$-\frac{1}{(W^\circ)^{n-1}}\frac{\partial U^\circ}{\partial x^\circ}\cdot FG''-\frac{1}{(W^\circ)^{n-2}}\frac{\partial\ln g}{\partial z^\circ}GG''-\frac{U^\circ}{(W^\circ)^{n-1}}\cdot\frac{\partial\ln g}{\partial x^\circ}FG''$$

$$=\frac{1}{g^{n+1}}\cdot\frac{\partial}{\partial\eta}\left\{\left[G''^2+\left(\frac{U}{W}F''\right)^2\right]^{\frac{n-1}{2}}\cdot G''\right\}+\frac{KL}{U_\infty^2(W^\circ)^n}\cdot f_{2z};$$

Their correct forms should be respectively.

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$$\frac{g^{n+1}}{(U^o)^{n-1}} \cdot \frac{\partial U^o}{\partial x^o}\left[F'^2 - FF'' - 1\right] + \frac{g^{n+1}\, W^o}{(U^o)^n} \cdot \frac{\partial U^o}{\partial z^o} \cdot \left[FG' - 1\right]$$

$$- \frac{g^{n+1}}{(U^o)^{n-1}} \cdot \frac{\partial W^o}{\partial z^o} \cdot GF'' - \frac{g^n}{(U^o)^{n-2}} \cdot \frac{\partial g}{\partial x^o} FF'' - \frac{g^n\, W^o}{(U^o)^{n-1}} \frac{\partial g}{\partial z^o} F''G$$

$$= (R)^{N(n+1)-1} \cdot \frac{\partial}{\partial \eta}\left\{\left[F''^2 + \left(\frac{W}{U}G''\right)^2\right]^{\frac{n-1}{2}} F''\right\} + \frac{k.L\; g^{n+1}}{U_\infty^2\,(U^o)^n} \cdot f_{2x} \quad ; \quad \textbf{(2.5.1)}$$

$$\frac{1}{(U^o)^{n-1}} \frac{\partial U^o}{\partial x^o}\left[F'^2 - FF'' - 1\right] + \frac{W^o}{(U^o)^{n-1}} \frac{\partial \ln U^o}{\partial z^o}\left[FG' - 1\right]$$

$$- \frac{1}{(U^o)^{n-1}} \cdot \frac{\partial W^o}{\partial z^o} \cdot GF'' - \frac{1}{(U^o)^{n-2}} \cdot \frac{\partial \ln g}{\partial x^o} \cdot FF'' - \frac{W^o}{(U^o)^{n-1}} \frac{\partial \ln g}{\partial z^o} F''G$$

$$= \frac{1}{g^{n+1}} \cdot \frac{\partial}{\partial \eta} \cdot \left\{\left[F''^2 + \left(\frac{W}{U}G''\right)^2\right]^{\frac{n-1}{2}} F''\right\} + \frac{k.L.}{U_\infty^2\,(U^o)^n} \cdot f_{2y} \quad \textbf{(2.5.2)}$$

and

$$\frac{1}{(W^o)^{n-1}} \cdot \frac{\partial W^o}{\partial z^o} \cdot \left[G'^2 - GG'' - 1\right] + \frac{U^o}{(W^o)^{n-1}} \cdot \frac{\partial \ln W^o}{\partial x^o}\left[FG' - 1\right]$$

$$- \frac{1}{(W^o)^{n-1}} \cdot \frac{\partial U^o}{\partial z^o} \cdot FG'' - \frac{1}{(W^o)^{n-2}} \frac{\partial \ln g}{\partial z^o} GG'' - \frac{U^o}{(W^o)^{n-1}} \frac{\partial \ln g}{\partial x^o} FG''$$

$$= \frac{1}{g^{n+1}} \cdot \frac{\partial}{\partial \eta}\left\{\left[G''^2 + \left(\frac{U}{W}G''\right)^2\right]^{\frac{n-1}{2}} G''\right\} + \frac{k.L}{U_\infty^2\,(W^o)^n} \cdot f_{2z} \quad \textbf{(2.5.3)}$$

Thus the modified form of the equation (17) of the above text is

$$(U^\circ)^{1-n} \cdot \frac{\partial U^\circ}{\partial x^\circ} = a_1 W^\circ (U^\circ)^{1-n} \frac{\partial \ln U^\circ}{\partial z^\circ} = a_2 (U^\circ)^{1-n} \frac{\partial W^\circ}{\partial z^\circ}$$

$$= a_3 (U^\circ)^{2-n} \frac{\partial \ln g}{\partial x^\circ} = a_4 (U^\circ)^{1-n} W^\circ \frac{\partial \ln g}{\partial z^\circ}$$

$$= a_5 \frac{1}{g^{n+1}} = a_6 (W^\circ)^{1-n} \cdot \frac{\partial W^\circ}{\partial z^\circ}$$

$$= a_7 U^\circ (W^\circ)^{1-n} \frac{\partial W^\circ}{\partial x^\circ} = a_8 (W^\circ)^{1-n} \cdot \frac{\partial U^\circ}{\partial x^\circ}$$

$$= a_9 (W^\circ)^{2-n} \frac{\partial \ln g}{\partial z^\circ} = a_{10} (W^\circ)^{1-n} . U^\circ \frac{\partial \ln g}{\partial x^\circ} \qquad (2.5.4)$$

The requirement of proportionality between U° and W° as obtained from (2.5.4) may be stated as

$$W^\circ = k_0 \, U^\circ \qquad (2.5.5)$$

and therefore (2.5.4)

$$(U^\circ)^{1-n} \frac{\partial U^\circ}{\partial x^\circ} = a_1 \cdot k_0 (U^\circ)^{1-n} \frac{\partial U^\circ}{\partial z^\circ} = a_2 \, k_0 (U^\circ)^{1-n} \frac{\partial U^\circ}{\partial z^\circ}$$

$$= a_3 \frac{(U^\circ)^{2-n}}{g} \cdot \frac{\partial g}{\partial x^\circ} = a_4 \, k_0 \frac{(U^\circ)^{2-n}}{g} \cdot \frac{\partial g}{\partial z^\circ}$$

$$= a_5 \frac{1}{g^{n+1}} = a_6 (k_0 U^\circ)^{1-n} . k_0 \frac{\partial U^\circ}{\partial z^\circ}$$

$$= a_7 (k_0 U^\circ)^{1-n} \cdot \frac{\partial U^\circ}{\partial x^\circ} = a_8 (k_0 U^\circ)^{1-n} \frac{\partial U^\circ}{\partial x^\circ}$$

$$= a_9 \frac{(k_0 U^\circ)^{2-n}}{g} \frac{\partial g}{\partial z^\circ} = a_{10} k_0^{1-n} \frac{(U^\circ)^{2-n}}{g} \frac{\partial g}{\partial x^\circ} \quad (2.5.6)$$

where k_0 is a constant, and

$$1 = a_7 k_0^{1-n} = a_8 k_0^{1-n} \; ; \quad a_1 = a_2 = a_6 k_0^{1-n}$$

$$a_3 = a_{10} k_0^{1-n} \; ; \quad a_4 = a_9 k_0^{1-n} \quad (2.5.7)$$

We also get

$$\frac{\partial \ln U^\circ}{\partial z^\circ} = a_1 \cdot k_0 \cdot \frac{\partial \ln U^\circ}{\partial x^\circ} = a_3 \frac{\partial \ln g}{\partial x^\circ}$$

$$= a_4 k_0 \frac{\partial \ln g}{\partial z^\circ} = a_5 \frac{(U^\circ)^{n-2}}{g^{n+1}} \quad (2.5.8)$$

Integrating (2.5.8) for U° and g separately, we get

$$U^\circ = f(x^\circ + A z^\circ) \quad (2.5.9)$$

$$g = [f(x^\circ + A z^\circ)]^p \quad (2.5.10)$$

where

$$1 = a_1 k_o A = a_3 p = a_4 \cdot k_o \cdot A \cdot p \qquad (2.5.11)$$

and

$$f' = a_5 f^{[(n-1) - p(n+1)]} \qquad (2.5.12)$$

Here A and p are arbitrary constants and f is an arbitrary function. The constant p corresponds to $\frac{\alpha-\beta}{\beta}$ of our discussions in the preceeding sections.

Integrating (2.5.12) and using (2.5.5), we get

$$U^o = c\,(x^o + Az^o)^m \qquad (2.5.13)$$

$$W^o = c\,k_o\,(x^o + Az^o)^m \qquad (2.5.14)$$

and

$$g = c^p\,(x^o + Az^o)^{pm} \qquad (2.5.15)$$

where $$m = \frac{1}{[p(n+1) - (n-2)]}$$

for the case when $p(n+1) - (n-2) \neq 0$ and $\alpha \neq 0$.

If $p(n+1) - (n-2) = 0$; we get the solutions

$$U^o = c\,e^{Bp.(x^o + Az^o)} \qquad (2.5.16)$$

$$W^{\circ} = k_{\circ}\, c\, e^{B p(x^{\circ}+Az^{\circ})} \tag{2.5.17}$$

$$q = c'\, e^{B(x^{\circ}+Az^{\circ})\, p\, \frac{n-2}{n+1}} \tag{2.5.18}$$

In the above equations, we can regard $k_{\circ}$, B, c and c' as arbitrary constants and then (2.5.17) and (2.5.11) would determine a_1 to a_{10} .

From (2.5.15), we find that the potential velocity is in a fixed direction and from (2.5.13) and (2.5.14) we find the ''similar solutions'' are possible when the potential velocity is proportional to some power of the distance from a fixed stagnationstraight line.

By putting $k_{\circ} = 0$ and $A = 0$, we get the two-dimensional case and the corresponding equations for

$p(n+1) - (n-2) \neq 0$ and $\alpha = 1$ reduce to

$$U^{\circ}(x^{\circ}) = c\,(x^{\circ})^{m} \tag{2.5.19}$$

which is the same as (2.3.8),

and $$g(x^{\circ}) = c^{p}\,(x^{\circ})^{\frac{1-2m+mn}{n+1}}$$

which again is the same as (2.3.9).

For $p(n+1) - (n-2) = 0$ equation (2.5.16) reduces to

$$U^{\circ}(x^{\circ}) = C\, e^{Bp x^{\circ}}$$
$$= C_1'\, e^{\left(-\frac{1-2m+mn}{n+1}\right) x^{\circ}} \qquad (2.5.20)$$

which again is the same as the value of $U(x)$ obtained in equation (2.3.13), also we get

$$g(x^{\circ}) = c'\, e^{Bp \frac{n-2}{n+1} x^{\circ}}$$
$$= C_1'\, e^{\frac{(n-2)(1-2m+mn)}{(n+1)^2} x^{\circ}} \qquad (2.5.21)$$

which is the same as (2.3.14) of our discussions. Thus we see that Schowalter's treatment gives result which are the same as ours after the mathematical error is corrected.

CHAPTER III

BOUNDARY-LAYER VELOCITY PROFILES FOR THE FLOW OF A POWER-LAW FLUID IN THE INLET LENGTH OF A CIRCULAR PIPE

3.1 INTRODUCTION

Bogue (1959) discussed the pressure drop in the inlet length of a circular pipe for the flow of the non-Newtonian power-law fluids. Following Kármán-Pohlhausen method, he assumed a cubic equation for the velocity profile. While three of the constants occuring in the cubic equation could be determined from the boundary conditions, the fourth constant 'a' was selected to make the integrated kinetic energy of the fully developed profile the same as that of the theoretical one as the excess pressure drop is mostly consumed in increasing the kinetic energy. This consideration gave 'a' $= \frac{n+1}{n}$ where n is the flow behaviour index of the

power-law fluid.

In the present chapter we have suggested and used an alternative principle viz. the principle of least squares for determining 'a'. We find that the estimates given by this method agree closely with those given by the principle of integrated kinetic energy.

The cubic velocity profile coincides with the fully developed velocity profile for $n = 1$ and $n = 1/2$, but for other values of n, it can be only a very good approximation. The question naturally arises whether a fourth degree profile will not give a better fit. But a fourth degree profile means one more unknown constant. Following the practice in boundary layer theory, we take the additional boundary condition as the vanishing of the second order derivative at the edge of the boundary layer. We get a velocity profile which coincides with the fully developed velocity profile for $n = 1/3$, $1/2$; but for other values of n, it can be only a good approximation. Even for $n = 1$, it does not coincide with the fully developed profile. For this reason, we investigate an alternative fourth degree velocity profile in which instead of the vanishing of the second order derivative, we insist that the velocity profile is such that it does reduce to the parabolic velocity profile for the Newtonian fluid.

We thus have three velocity profiles, one of the third degree and two of the fourth degree, each of these

containing an unknown constant 'a'. To determine each 'a', we have three principles viz.

(i) $\frac{n+1}{n}$

(ii) the least squares principle

(iii) the principle of integrated kinetic energy.

We investigate all these possibilities in the present chapter. Another fourth degree velocity profile is obtained by combining the principles of equal integrated kinetic energy and the principle of least squares. We determine two constants occuring in this equation from the boundary conditions and the other two can, in principle, be determined from the two principles. In practice, however, we obtain two cubic equations to determine the two constants 'a' and 'd', as it is not easy to obtain the solution of these two equations.

3.2 VELOCITY PROFILES

(i) Cubic Velocity Profile

Let the boundary layer velocity profile in the inlet length of a circular pipe be given by

$$\frac{u}{U} = a\frac{(1-r/D)}{\delta/D} + b\frac{(1-r/D)^2}{\delta^2/D^2} + c\frac{(1-r/D)^3}{\delta^3/D^3} \qquad (3.2.1)$$

where

u = the velocity component parallel to the x-axis on the section at a distance x from the origin

and at a distance r from the central axis of the pipe.

U = the velocity in the central plug

$2D$ = the diameter of the pipe

δ = the boundary layer thickness and is a function of x only.

The constants, a, b and c occuring in (3.2.1) are determined from the boundary conditions :

$$\begin{aligned} u &= U \quad \text{at} \quad r = D-\delta \\ \frac{\partial u}{\partial r} &= 0 \quad \text{at} \quad r = D-\delta \end{aligned} \qquad (3.2.2)$$

These give the values of 'b' and 'c' in terms of 'a', so that the resulting velocity profile is of the form :

$$\frac{u}{U} = a\,\frac{(1-r/D)}{\delta/D} + (3-2a)\,\frac{(1-r/D)^2}{\delta^2/D^2} + (a-2)\,\frac{(1-r/D)^3}{\delta^3/D^3} \qquad (3.2.3)$$

(ii) Fourth degree velocity profile No. 1

We assume the velocity profile in the form

$$\frac{u}{U} = a\,\frac{(1-r/D)}{\delta/D} + b\,\frac{(1-r/D)^2}{\delta^2/D^2} + c\,\frac{(1-r/D)^3}{\delta^3/D^3} + d\,\frac{(1-r/D)^4}{\delta^4/D^4} \qquad (3.2.4)$$

The constants b, c and d are determined by the boundary conditions :

$$u = U \quad \text{at} \quad r = D-\delta$$

$$\frac{\partial u}{\partial r} = 0 \quad \text{at} \quad r = D-\delta \qquad (3.2.5)$$

$$\frac{\partial^2 u}{\partial r^2} = 0 \quad \text{at} \quad r = D-\delta$$

On applying these conditions (3.2.4) becomes :

$$\frac{u}{U} = a\frac{(1-r/D)}{\delta/D} + (6-3a)\frac{(1-r/D)^2}{\delta^2/D^2} + (3a-8)\frac{(1-r/D)^3}{\delta^3/D^3} + (3-a)\frac{(1-r/D)^4}{\delta^4/D^4} \qquad (3.2.6)$$

(iii) Fourth degree velocity profile No. 2

We feel it advisable to assume the fourth degree profile which must reduce to the second degree profile for the Newtonian case. This requires in addition to the boundary conditions (3.2.2) that

$$c = d = f(a) \qquad (3.2.7)$$

This gives,

$$\frac{u}{U} = a\frac{(1-r/D)}{\delta/D} + \frac{(7-5a)}{3}\frac{(1-r/D)^2}{\delta^2/D^2} + \frac{(a-2)}{3}\frac{(1-r/D)^3}{\delta^3/D^3} + \frac{(a-2)}{3}\frac{(1-r/D)^4}{\delta^4/D^4}$$

(3.2.8)

(iv) Fully developed velocity profile

The fully developed velocity profile for the flow of a power-law fluid in a circular pipe (see Wilkinson (1960)) is

$$\frac{u}{V} = \frac{(3n+1)}{(n+1)}\left[1 - (r/D)^{\frac{n+1}{n}}\right] \qquad (3.2.9)$$

where V is the entrance velocity. Since the flux of the liquid passing through the pipe remains constant, the equation of continuity at any section of the pipe is given by

$$\pi D^2 V = \pi (D-\delta)^2 . U + 2\pi \int_{D-\delta}^{D} r u \, dr \qquad (3.2.10)$$

giving

$$\frac{U}{V} = \frac{1}{1 + K_1 \delta/D + K_2 \delta^2/D^2} \qquad (3.2.11)$$

The values of the denominator of (3.2.11) when $\delta = D$ for various velocity profiles are as follows :

For the third degree velocity profile given by the equation

(3.2.3), it is $= \frac{a+3}{10}$ (3.2.12)

For the fourth degree velocity profile given by the equation

(3.2.6), it is $= \frac{a+6}{15}$ (3.2.13)

For the fourth degree velocity profile given by the equation

(3.2.8), it is $= \frac{2a+5}{18}$ (3.2.14)

3.3 LEAST SQUARES PRINCIPLE OF FINDING ''a''

Since the assumed velocity profiles (3.2.3),(3.2.6) and (3.2.8) determined with the help of the equations (3.2.12), (3.2.13) and (3.2.14) at the inlet length $\delta = D$ in the limit should approach as near to the fully developed velocity profile (3.2.9) as possible, we apply the principle of least squares to choose 'a' so that

$$\int_0^D (u-u_1)^2 \, dr \tag{3.3.1}$$

is minimised. Here u is the fully developed velocity given by (3.2.9) and (3.2.11), and u_1 is the velocity given by (3.2.3) or (3.2.6) or (3.2.8).

Using this we get

(i) for the cubic velocity profile (3.2.3)

$$a = \frac{3(600n^3+1494n^2+585n+57)}{(2440n^3+1394n^2+251n+19)} \tag{3.3.2}$$

(ii) for the fourth degree velocity profile (3.2.6)

$$a = \frac{2(180n^3+5196n^2+2055n+171)}{(5340n^3+2948n^2+485n+33)} \tag{3.3.3}$$

(iii) for the fourth degree velocity profile (3.2.8)

$$a = \frac{(232560n^4+532692n^3+268916n^2+48953n+2999)}{2(139680n^4+101376n^3+26900n^2+3398n+176)} \tag{3.3.4}$$

The following table gives values of ' a ' for various values of n for all the four cases viz. (3.3.2), (3.3.3), (3.3.4) and that assumed by Bogue (1959).

Table 3.1 : Values of 'a' for various values of n for all velocity profiles

n	'a' for cubic velocity profile (3.2.3)	'a' for fourth degree velocity profile (3.2.6)	'a' for fourth degree velocity profile (3.2.8)	$a = \frac{n+1}{n}$
0.6	2.6920	2.6103	2.6434	2.6666
0.7	2.4571	2.3117	2.4248	2.4286
0.8	2.2722	2.0757	2.2528	2.2500
0.9	2.1229	1.8845	2.1141	2.1111
1.0	2.0000	1.7266	2.0000	2.0000
1.1	1.8970	1.5938	1.9044	1.9091
1.2	1.8096	1.4808	1.8232	1.8333
1.3	1.7343	1.3833	1.7534	1.7692
1.4	1.6689	1.2984	1.6928	1.7143
1.5	1.6115	1.2228	1.6396	1.6667

The above table shows that $a = \frac{n+1}{n}$ gives a good approximation for cubic velocity profile as well as for the second fourth degree velocity profile, but the agreements breakdown for the first fourth degree velocity profile. A possible reason may be that the second fourth degree velocity

profile is quite close to the third degree velocity profile as both of them are assumed to be reducible to the parabolic velocity profile for Newtonian case.

4.4 INTEGRATED KINETIC ENERGY PRINCIPLE FOR FINDING 'a'.

To make the two velocity profiles, viz. the fully developed velocity profile and the limiting velocity profile correspond to each other, it is evident, however, that their integrated kinetic energies must also be the same. Hence we choose 'a' so that

$$\frac{1}{2}\int_0^D r u^3 dr = \frac{1}{2}\int_0^D r u_1^3 dr \qquad (3.4.1)$$

where u is the fully developed velocity given by (3.2.9) and u_1 is the limiting theoretical velocity given by either (3.2.3) or (3.2.6) or (3.2.8).

Equation (3.4.1) gives :

(i) for the cubic velocity profile (3.2.3)

$$\frac{3(3n+1)^2}{2(2n+1)(5n+3)} = \left(\frac{10}{a+3}\right)^3 \left(\frac{7a^3+46a^2+183a+549}{9240}\right) \qquad (3.4.2)$$

(ii) for the fourth degree velocity profile (3.2.6)

$$\frac{3(3n+1)^2}{2(2n+1)(5n+3)} = \left(\frac{15}{a+6}\right)^3 \left(\frac{30a^3+311a^2+2056a+12336}{120120}\right) \qquad (3.4.3)$$

(iii) for the fourth degree velocity profile (3.2.8)

$$\frac{3(3n+1)^2}{2(2n+1)(5n+3)} = \left(\frac{3}{2a+5}\right)^3 \left(\frac{10158a^3 - 42661a^2 + 397010a + 511575}{45045}\right) \qquad (3.4.4)$$

Each of these equations (3.4.2), (3.4.3) and (3.4.4) being cubic in 'a' can be solved for 'a', but instead of solving for 'a' we choose 'a' for convenient values of n and with this value of 'a' we compare the degree of agreement of the method of least squares or with the case of $a = \frac{n+1}{n}$. If R denotes the ratio of integrated kinetic energies between the assumed velocity profile and the fully developed velocity profile in each of the three velocity profiles the comparison is shown in the following table :

(i) for the cubic velocity profile given by (3.2.3)

$$R = \left(\frac{10}{a+3}\right)^3 \frac{\dfrac{7a^3 + 46a^2 + 183a + 549}{9040}}{\dfrac{3(3n+1)^2}{2(2n+1)(5n+3)}} \qquad (3.4.6)$$

Table 3.2 : Comparison with all the values of 'a' for equation (3.4.7)

n	R when $a = \frac{n+1}{n}$	R when 'a' is given by least squares method	R when 'a' is given by equal integrated K.E.
0.6	0.9992876	0.9956860	1.0000
0.7	0.9993416	0.9947159	1.0000
0.8	0.9995935	0.9956053	1.0000
0.9	0.9998364	0.9975257	1.0000
1.0	1.0000000	1.0000000	1.0000
1.1	1.0000739	1.0027474	1.0000
1.2	1.0000645	1.0056005	1.0000
1.3	0.9999853	1.0084589	1.0000
1.4	0.9998502	1.0112632	1.0000
1.5	0.9996722	1.0139794	1.0000

(ii) for the fourth degree velocity profile (3.2.6)

$$R = \left(\frac{15}{a+6}\right)^3 \frac{\dfrac{30a^3 + 311a^2 + 2056a + 12336}{120120}}{\dfrac{3(3n+1)^2}{2(2n+1)(5n+3)}} \qquad (3.4.6)$$

Table 3.3 : Comparison with all the values of 'a' for equation (3.4.6)

n	R, when $a = \frac{n+1}{n}$	R when 'a' is given by least squares method	R when 'a' is given by equal integrated K.E.
0.6	0.9979694	1.0065536	1.000000
0.7	0.9949700	1.0126665	1.000000
0.8	0.9914674	1.0197556	1.000000
0.9	0.9877684	1.0268058	1.000000
1.0	0.9840501	1.0335075	1.000000
1.1	0.9804236	1.0397818	1.000000
1.2	0.9769436	1.0456102	1.000000
1.3	0.9736377	1.0510042	1.000000
1.4	0.9705160	1.0560354	1.000000
1.5	0.9675790	1.0605264	1.000000

(iii) for the fourth degree velocity profile (3.2.8)

$$R = \left(\frac{3}{2a+5}\right)^3 \frac{\dfrac{10158a^3 - 42661a^2 + 397010a + 511575}{45045}}{\dfrac{3(3n+1)^2}{2(2n+1)(5n+3)}} \tag{3.4.7}$$

Table 3.4 : Comparison with all the values of 'a' for equation (3.4.7)

n	R when $a = \frac{n+1}{n}$	R when 'a' is given by least squares method	R when 'a' is given by equal integrated K.E.
0.6	0.8890685	0.8965474	1.0000
0.7	0.927[illegible]44	0.9588164	1.0000
0.8	0.9672834	0.9518108	1.0000
0.9	0.9809138	0.9798082	1.0000
1.0	1.000000	1.000000	1.0000
1.1	1.0166127	1.0174631	1.0000
1.2	1.0285297	1.0326670	1.0000
1.3	1.0393562	1.0459307	1.0000
1.4	1.0485040	1.0517399	1.0000
1.5	1.0563061	1.0681618	1.0000

We observe that almost in all cases, the agreement between the three methods is quite good. Our discussion, therefore justifies to some extent, Bogue's choice of $\frac{n+1}{n}$ for a . We may however, use other more exact values if we like.

CHAPTER IV

AXIALLY-SYMMETRIC AND TWO-DIMENSIONAL STAGNATION POINT FLOW OF A CERTAIN VISCO-ELASTIC FLUID

4.1 INTRODUCTION

The problem of viscous flow near a stagnation point had been considered by Howarth (1934) for the plane case and by Frossling for the axially-symmetric case as has been reported by Schlichting (1960) and Pai (1956). For the Reiner-Rivlin fluid with constant coefficients of viscosity and cross-viscosity, the problem was solved by Srivastava, A.C. (1958) by using the Karman-Pohlhausen method employing a fourth degree velocity profile. Later Jain (1961) integrated the same basic differential equation by extremal point collocation method on IBM 650 computer at Hamberg. The results of Srivastava and Jain regarding

the increase in boundary layer thickness and skin friction were almost reverse of each other. Again Rajeshwari and Rathna (1962) integrated the equations for a more general non-Newtonian viscoelastic fluid which included the Reiner-Rivlin fluid as a particular case, by using the sixth degree velocity profile and Karman-Pohlhausen method. The results agreed with the results of Jain. This shows that there was nothing wrong with the use of Karman-Pohlhausen method as an approximate method of integration.

A close examination shows that the use of a fourth degree profile by Srivastava is not justified by the boundary conditions imposed. In fact, he uses the vanishing of the third derivative of a certain function at the edge of the boundary layer to determine a constant of integration occuring in his problem but does not use the same condition for determining the constants in the velocity profile. If this condition had been used, the boundary layer thickness could have been determined without integrating the basic differential equation. If all the conditions have to be satisfied consistently, the use of sixth degree velocity profile is imperative.

Sharma (1959a) has investigated the axially-symmetric flow of a single relaxation-time viscoelastic fluid near a stagnation point by using a fourth degree velocity profile and for the same reasons mentioned above the use of a sixth degree profile is necessary in this case

also. We hereby give the results of the use of the sixth degree profile in section 4.3 . We find in this that the boundary layer thickness and the skin friction both decrease with the increase of the non-dimensional viscoelastic parameter. The two-dimensional stagnation point flow has also been considered for our model of the fluid and the result comes out to be the same as that of Rajeshwari and Rathna (1962) for a different model except for the change in the sign of the elastic parameter. As such the detailed calculations for this case have not been given.

4.2 DISCUSSION OF THE BOUNDARY CONDITIONS

Let a visco-elastic liquid impinge perpendicularly on the wall at $z = 0$ and flow away radially in all directions, so that the stagnation point is at the origin of coordinates. Let U, W be the velocity components in the r and z directions in the frictionless flow, so that

$$U = ar \quad , \quad W = -2az \tag{4.2.1}$$

a being a constant. In the boundary layer the respective velocity components are given by

$$u = r f'(z) \quad , \quad w = -2f(z) \tag{4.2.2}$$

substituting these in the equations of motion, Sharma (1959a) gets the equation

$$\frac{\partial}{\partial z}\left[f'^2 - 2ff'' - \nu f''' + \nu\lambda^* \left(4f'f''' - 2ff^{iv}\right) \right] = 0 \qquad (4.2.3)$$

where ν is the kinematic coefficient of viscosity and λ^* is the relaxation time parameter and prime denotes differentiation with respect to z . Integration of the above equation gives

$$2ff'' - f'^2 + \nu f''' - \nu\lambda^* \left(4f'f''' - 2ff^{iv}\right) = c \qquad (4.2.4)$$

where c is the constant of integration.

Boundary conditions used are

$$\begin{aligned} f(0) &= f'(0) = 0 \\ f'(\delta^*) &= a \;, \quad f''(\delta^*) = 0 \qquad (4.2.5) \\ f'''(\delta^*) &= \lambda^* \left[4f'''(\delta^*) \,.\, a - 2f(\delta^*)\, f^{iv}(\delta^*) \right] \end{aligned}$$

where δ^* is the boundary layer thickness.

The first two of the conditions (4.2.5) come from no-slip condition of the liquid, the next two came from the physical assumptions that $u = U$ and $\frac{\partial u}{\partial z} = 0$ at the edge of the boundary layer.

The last condition does not have any physical basis, though it determines $c = -a^2$. Substituting this value and the first two boundary conditions in the differential equation (4.2.4) at $z = 0$, we get another condition

viz.

$$f'''(0) = -a^2/\gamma \tag{4.2.6}$$

The last of the conditions (4.3.5) has not been used by Sharma (1959a) to determine the constants in the velocity profile. Substituting

$$\delta = \sqrt{\frac{a}{\gamma}} \cdot \delta^* \quad ; \quad \eta = \sqrt{\frac{a}{\nu}} \cdot \frac{z}{\delta}$$

$$f(z) = \sqrt{a\nu}\ \phi(\eta) \quad ; \quad \lambda = a\lambda^* \tag{4.2.7}$$

the basic differential equation and the boundary conditions become :

$$\delta\left(2\delta\phi\phi'' - \delta\phi'^2 + \phi''' + \delta^3\right) - \lambda\left(4\phi'\phi''' - 2\phi\phi^{iv}\right) = 0 \tag{4.2.8}$$

$$\phi(0) = \phi'(0) = 0$$

$$\phi'(1) = \delta \quad ; \quad \phi''(1) = 0 \quad ; \quad \phi'''(0) = \delta^3 \tag{4.2.9}$$

and
$$\delta\phi'''(1) - \lambda\left[4\delta\phi'''(1) - 2\phi(1)\ \phi^{iv}(1)\right] = 0$$

By using the first five of the above conditions here, the boundary layer velocity profile becomes :

$$\phi(\eta) = \frac{\delta^3 + 6\delta^2}{8}\eta^2 - \frac{\delta^3}{6}\eta^3 - \frac{2\delta - \delta^3}{16}\eta^4 \tag{4.2.10}$$

If we use the last condition also, we get

$$8(\delta^3 - 6\delta) = \lambda(\delta^6 + 60\delta^4 - 252\delta^2) \quad (4.2.11)$$

which will determine the boundary layer thickness in terms of the parameter λ, and we would get the result even without integrating the basic differential equation. This is, of course, not a satisfactory position. This difficulty can be overcome by using the boundary conditions :

$$\phi(0) = \phi'(0) = 0$$

$$\phi'(1) = \delta, \ \phi''(1) = 0, \ \phi'''(1) = 0 \text{ and } \phi^{iv}(1) = 0 \quad (4.2.12)$$

Also from the differential equation (4.2.8), we get

$$\phi'''(0) = -\delta^3$$

and thus we would in all get seven boundary conditions which will be sufficient to determine all the seven constants in the sixth degree profile

$$\phi(\eta) = a_0 + a_1\eta + a_2\eta^2 + a_3\eta^3 + a_4\eta^4 + a_5\eta^5 + a_6\eta^6 \quad (4.2.13)$$

It is obvious, that the velocity profile of degree less than six would not do, though we can use profiles of higher degree which would, however, mean more complicated

calculations.

We may also note that the above boundary conditions are the ones we usually use when a profile of degree six is used.

4.3 DETERMINATION OF BOUNDARY-LAYER THICKNESS AND SKIN-FRICTION

Using the boundary conditions (4.2.12), (4.2.13) is obtained as

$$\phi(\eta) = \frac{\delta^3+20\delta}{16}\eta^2 - \frac{\delta^3}{6}\eta^3 + \frac{3\delta^3-20\delta}{16}\eta^4 - \frac{\delta^3-10\delta}{10}\eta^5 + \frac{\delta^3-12\delta}{48}\eta^6 \qquad (4.3.1)$$

Integrating the differential equation (4.2.8) between the limits 0 and 1, we have

$$2\delta^2\left[\phi\phi'\right]_0^1 - 3\delta^2\int_0^1 \phi'\,d\eta + \delta\left[\phi''\right]_0^1 - 4\lambda\left[\phi\phi'''\right]_0^1 + 6\lambda\int_0^1 \phi\,\phi^{iv}\,d\eta = 0 \qquad (4.3.2)$$

Substituting the values of $\phi(\eta)$ and its derivatives from (4.3.1), simplifying and putting $\delta^2 = x$, we get

$$7x^3 + (448-440\lambda)x^2 - (31640+13200\lambda)x + (184800-792000\lambda) = 0 \qquad (4.3.3)$$

Since $x \neq 0$, the above equation reduces to

$$x^3 + 64x^2 - 4520x + 26400 = 0 \tag{4.3.4}$$

for the case $\lambda = 0$ i.e. for non-elastic Newtonian viscous liquid. This is the same equation as obtained by Rajeshwari and Rathna (1962) for the same case.

This equation (4.3.4) has two roots 6.4994 and 37.58115 and a negative root. The velocity profile corresponding to 6.4994 closely agrees with that of Frossling as reported by Schlichting (1960). The roots of the equation (4.3.3) for various values of λ are given in the following table :

Table 4.1 : Roots of the equation (4.3.3) for various values of λ .

λ	$\delta^2 = x$	δ	δ^3	$\phi''(0)$
.000	6.4994	2.5494	16.5696	8.4447
.001	6.4637	2.5424	16.4333	8.4101
.002	6.4280	2.5353	16.2969	8.3753
.004	6.3566	2.5212	16.0262	8.3063
.006	6.2859	2.5071	15.7594	8.2377
.008	6.2147	2.4929	15.4926	8.1698
.010	6,1454	2.4789	15.2338	8.1015
.020	5.8021	2.4087	13.9765	7.7687
.100	3.3494	1.8301	6.1297	5.3415
.150	3.2697	1.8082	5.9123	5.2595
.200	3.1943	1.7873	5.7092	5.1819

The shear at the wall $z = 0$ is given by

$$t_{rz}\Big|_{z=0} = \rho \frac{a^{3/2}}{\sqrt{\nu}} \cdot r \cdot \Phi''(0) \qquad (4.3.5)$$

The above table shows :

(i) the boundary layer thickness decreases with the increase in λ ;

(ii) the shearing stress at the wall also decreases with the increase in λ .

Sharma (1959a) has found that both δ and $\Phi''(0)$ increase with λ . The reason is not so much in the degree of the profile used as with the slight error with the cubic equation which should have been :

$$3x^3 + (44 - 119\lambda)x^2 - (912 + 1344\lambda)x$$

$$+ (2520 - 5796\lambda) = 0 \qquad (4.3.6)$$

If the correct equation is taken the results would have been the same as ours.

4.4 EQUATION OF STREAM LINES

The equation of stream lines is

$$r^2 \Phi(\eta) = \text{constant} \qquad (4.4.1)$$

or

$$r^2 F(z) = \text{constant}$$

where

$$F(Z) = \left[\frac{\delta^2+20}{16\delta} Z^2 - \frac{Z^3}{6} + \frac{3\delta^2-20}{16\delta^3} Z^4 \right.$$

$$\left. - \frac{\delta^2-10}{10\delta^4} Z^5 + \frac{\delta^2-12}{48\delta^5} Z^6 \right] \quad \textbf{(4.4.2)}$$

and

$$Z = \eta\delta = \sqrt{\frac{a}{\nu}} \cdot z$$

Table 4.2 gives the values of $F(Z)$ as against Z .

Table 4.2 : Values of $F(Z)$ for different values of Z and λ

Z	$\lambda = 0$	$\lambda = .01$	$\lambda = .15$
0.2	0.0246	0.0251	0.0301
0.5	0.1417	0.1437	0.1745
1.0	0.4883	0.4950	0.5253
1.5	0.9404	0.9634	1.0447
2.0	1.4335	1.4549	1.5306
δ	1.9811	1.9226	1.8808

4.5 TWO-DIMENSIONAL STAGNATION POINT FLOW

The results of the boundary layer thickness are apparently the reverse of the results of Rajeshwari and Rathna (1962) but the models of the visco-elastic fluid

are also different in two cases and there is no apriori reason why this should give the same results. However, to investigate this point further we study in this section the two-dimensional stagnation point flow of the same fluid. We take

$$u = x f'(y) \quad , \quad v = - f(y) \tag{4.5.1}$$

so that following the earlier method, stress tensor components upto the first power of $\mu\lambda^*$ are

$$\begin{aligned} t_{xx} &= 2\mu f' + \mu\lambda^* \left[4f'^2 + 2ff'' + 2x^2 f''^2\right] \\ t_{xy} &= \mu x \left[f'' + \lambda^* (ff''' - 3f'f'') \right] \\ t_{yy} &= \mu \left[-2f' + \lambda^* (4f'^2 - 2ff'') \right] \end{aligned} \tag{4.5.2}$$

Substituting these in the equations of motion and eliminating the pressure, the condition of integrability comes out to be :

$$x \frac{\partial}{\partial y} \left[(ff'' - f'^2) + \nu \left\{ f''' + \lambda^* (f f^{IV} - 2f'f''' + f''^2) \right\} \right] = 0 \tag{4.5.3}$$

Making the substitutions in terms of non-dimensional variables and parameters :

$$\eta = \sqrt{\frac{a}{\nu}} \cdot \frac{y}{\delta} \quad \text{where} \quad \delta = \sqrt{\frac{a}{\nu}}\, \delta^*$$

$$f(y) = \sqrt{a\nu} \cdot \phi(\eta) \tag{4.5.4}$$

and $$\lambda = a\lambda^*$$

The equations (4.6.3) reduce to

$$\delta\left[\delta\phi'^2 - \delta\phi\phi'' - \phi''' - \delta^3\right] - \lambda\left[\phi\phi^{iv} - 2\phi'\phi'' + \phi''^2\right] = 0 \quad (4.6.5)$$

which is the same as equation (2.3) of Rajeshwari and Rathna (1962) except that the sign of λ has changed. The boundary conditions are also the same. Thus in this case the effects of the parameter is exactly the same as that of $-\lambda$ in the discussions of Rajeshwari and Rathna (1962). We are, therefore, not surprised that the effects of λ are in opposite direction in the two models.

Again the dissipation function for our model comes out to be

$$\frac{\Phi}{\mu a^2} = \frac{1}{\delta^2}\left[4\phi'^2 + \xi^2\phi''\right] + \frac{\lambda}{\delta^3}\left[\rho(\phi\phi^{iv} - \phi'\phi'')\phi'' + 4\phi'\phi''\right] \quad (4.6.6)$$

where

$$\xi = \frac{x}{\delta}\sqrt{\frac{a}{\nu}}$$

which is again exactly the same as the corresponding expression obtained by Rajeshwari and Rathna (1962) except for the change of the sign of λ.

To determine the reasons for this difference, let t'_{xx}, t'_{xy} and t'_{yy} denote the contributions of the elastic

term to the model of the fluid used by Rajeshwari and Rathna (1962) and t''_{xx} , t''_{xy} and t''_{yy} the corresponding contributions for our model, then we find that

$$\begin{aligned} t''_{xx} &= t'_{yy} \\ t''_{yy} &= t'_{xx} \end{aligned} \qquad (4.5.7)$$

and

$$t''_{xy} = - t'_{xy}$$

The contribution of these terms to the integrability equation of our model are

$$\frac{\partial}{\partial y}\left[\frac{\partial}{\partial x}(t''_{xx}) + \frac{\partial}{\partial y}(t''_{xy})\right] - \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}(t''_{xy}) + \frac{\partial}{\partial y}(t''_{yy})\right] \qquad (4.5.8)$$

which reduces to

$$\frac{\partial}{\partial y}\left[\frac{\partial}{\partial x}(t'_{yy}) + \frac{\partial}{\partial y}(-t'_{xy})\right] - \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}(-t'_{xy}) + \frac{\partial}{\partial y}(t'_{xx})\right] \qquad (4.5.9)$$

and this is negative of the corresponding contribution of the model used by Rajeshwari and Rathna (1962). Similarly the contributions by the elastic term to the dissipative function are easily seen to be of equal magnitude but of opposite sign.

Both models taken by Rajeshwari and Rathna (1962) and our model (when we neglect higher powers of λ^*) are

approximation of the more general visco-elastic fluid models, and we find that they agree upto this order of approximation except for the change of sign of the parameter used. The agreement is exact for the two-dimensional case and is very good for the axially-symmetric case.

CHAPTER V

ON STEADY MOTION OF A VISCO-ELASTIC LIQUID BETWEEN TWO PLANE BOUNDARIES AND IN AN ANNULUS WITH AND WITHOUT SUCTION

5.1 INTRODUCTION

In this chapter we consider a certain type of visco-elastic liquid whose constitutive equation is a modification of the Oldroyd's equation (1.2.3) and is denoted as

$$t_{ij} + \lambda \dot{t}_{ij} = 2\mu e_{ij} \qquad (5.1.1)$$

where t_{ij} is the stress tensor, e_{ij} the strain-rate tensor, $\dot{t}_{ij}$ is the rate of stress tensor and is equal to the expression

$$\frac{\partial}{\partial t} t_{ij} + t_{ij,k} v_k - t_{ik} v_{j,k} - t_{kj} v_{i,k} + v_{k,k} \qquad (5.1.2)$$

where ϑ_i are the velocity components, λ is the relaxation time parameter and μ is a scalar coorsponding to the viscosity coefficient. The flow of such a fluid in an annulus between coaxial circular cylinders and between two parallel plane boundaries has been considered by Datta (1960). He assumed that the fluid is injected at one boundary and withdrawn at the other both at the same rate. He further assumed one of the boundaries to be moving with a constant velocity parallel to the axis. He also neglected the squares and higher powers of the relaxation time parameter.

We have found it possible to obtain solutions for all values of the relaxation time parameter λ provided the velocity of suction and injection is small. We have also found the solutions for the case when the velocity of suction and injection has any value but the relaxation time parameter is small. The second case has also been considered by Datta (1960), but his equation (10) does not include the second order derivative. We have obtained a series solution for this case.

We have derived a number of properties of the flows from our solutions. Further, tables have been prepared to illustrate the results quantitatively.

PART ONE

FLOW BETWEEN PARALLEL PLATES

5.2 BASIC EQUATIONS AND ANALYTIC SOLUTION

We choose the rectangular Cartesian coordinates with x-axis along the upper plate which is fixed and y-axis perpendicular to the plates directed into the liquid. The flow is assumed to be in the x-direction due to the motion of the lower plate and the liquid is injected at the lower plate and withdrawn at the upper plate (or it may be injected at the upper plate and withdrawn at the lower plate). Then if we assume the flow to be steady and fully developed and the velocity components to be the functions of y alone, we have :

$$U_x = u(y) \quad , \quad v_y = v(y) \tag{5.2.1}$$

The boundary conditions are

$$\begin{aligned} u &= 0 && \text{at} && y = 0 \\ &= U && \text{at} && y = h \\ v &= -V && \text{at} && y = 0 \\ &= -V && \text{at} && y = h \end{aligned} \tag{5.2.2}$$

together with the equation of continuity

$$v = -V \qquad \text{everywhere} \tag{5.2.3}$$

The stress strain-rate relations are :

$$t_{xx} + \lambda \left[v \frac{\partial t_{xx}}{\partial y} - 2 \frac{\partial u}{\partial y} \cdot t_{xy} \right] = 0$$

$$t_{yy} + \lambda \left[v \frac{\partial t_{yy}}{\partial y} \right] = 0 \tag{5.2.4}$$

$$t_{xy} + \lambda \left[v \frac{\partial t_{xy}}{\partial y} - \frac{\partial u}{\partial y} \cdot t_{yy} \right] = \mu \left(\frac{\partial u}{\partial y} \right)$$

The equations of momentum give :

$$\rho v \frac{\partial u}{\partial y} = \frac{\partial t_{xy}}{\partial y}$$

$$0 = \frac{\partial p}{\partial y} + \frac{\partial t_{yy}}{\partial y} \tag{5.2.5}$$

Eliminating t_{yy} from second and the third equations of (5.2.4) we get

$$t_{xy} \left(\frac{\partial u}{\partial y} - v\lambda \frac{\partial^2 u}{\partial y^2} \right) + 2 v \lambda \frac{\partial u}{\partial y} \cdot \frac{\partial t_{xy}}{\partial y} = \mu \left(\frac{\partial u}{\partial y} \right)^2 \tag{5.2.6}$$

This is different from Dattas' equation no. (18) and it does not reduce to his equation by neglecting λ^2 .

From the first of the equations (5.2.5) on integrating and applying the boundary condition $v = -V$, we get :

$$t_{xy} = -\rho u V + D , \tag{5.2.7}$$

where D is the constant of integration.

Eliminating t_{xy} from (5.2.5) and (5.2.6) we get a second order differential equation which can at once be integrated since it does not contain y explicitly. Alternatively solving the second equation of (5.2.3) we get

$$t_{yy} = A e^{y/\lambda V} \tag{5.2.8}$$

where A is a constant.

We can, obtain solutions for two extreme cases viz when λ is very small or when V is very small. This we proceed to do in the succeeding sections.

5.3 SERIES SOLUTION FOR SMALL VELOCITIES OF SUCTION AND INJECTION AND FOR ALL RELAXATION TIME PARAMETER

Assuming the series solution for (5.2.5) in the form :

$$u = \sum_{m=0}^{M} u_m V^m$$

along with (5.3.1)

$$D = \sum_{m=0}^{M} D_m V^m$$

and substituting (5.2.7) and (5.3.1) in (5.2.6) we get the first few terms of the differential equation as :

$$\Big[-\rho V\left(u_0+Vu_1+V^2u_2+V^3u_3+V^4u_4+V^5u_5+\cdots\right)+\left(D_0+VD_1+V^2D_2\right.$$

$$\left.+V^3D_3+V^4D_4+V^5D_5+\cdots\right)\Big]\times\Big[\frac{\partial u_0}{\partial y}+V\frac{\partial u_1}{\partial y}+V^2\frac{\partial u_2}{\partial y}+V^3\frac{\partial u_3}{\partial y}$$

$$+V^4\frac{\partial u_4}{\partial y}+V^5\frac{\partial u_5}{\partial y}+\cdots\Big]+2V^2\lambda\rho\Big[\frac{\partial u_0}{\partial y}+V\frac{\partial u_1}{\partial y}+V^2\frac{\partial u_2}{\partial y}+V^3\frac{\partial u_3}{\partial y}$$

$$+V^4\frac{\partial u_4}{\partial y}+V^5\frac{\partial u_5}{\partial y}+\cdots\Big]^2+\lambda\Big[V\frac{\partial^2u_0}{\partial y^2}+V^2\frac{\partial^2u_1}{\partial y^2}+V^3\frac{\partial^2u_2}{\partial y^2}+V^4\frac{\partial^2u_3}{\partial y^2}\Big]\times$$

$$\Big[-\rho V\left(u_0+u_1V+V^2u_2+V^3u_3+V^4u_4+\cdots\right)+\left(D_0+VD_1+V^2D_2+V^3D_3\right.$$

$$\left.+V^4D_4+V^5D_5+\cdots\right)\Big]=\mu\Big[\frac{\partial u_0}{\partial y}+V\frac{\partial u_1}{\partial y}+V^2\frac{\partial u_2}{\partial y}+\cdots\Big]^2 \quad (5.3.2)$$

Collecting the terms independent of V i.e. for the case of no suction or injection we have

$$\frac{\partial u_0}{\partial z}=\frac{D_0 h}{\mu} \quad (5.3.3)$$

where

$$z=\frac{y}{h}$$

Integrating this equation and applying the boundary conditions :

$$\begin{aligned} u_0 &= U \quad \text{at} \quad z=1 \\ &= 0 \quad \text{at} \quad z=0 \end{aligned} \quad (5.3.4)$$

we get

$$\frac{u_0}{U} = z \qquad (5.3.5)$$

and

$$D_0 = \frac{U\mu}{h}$$

This result agrees with the result obtained by Datta (1960) for the case of no suction or injection.

When the squares and higher powers of V are neglected, the coefficients of V from (5.3.2) give the linear differential equation

$$\frac{\partial u_1}{\partial z} + \frac{\rho U h}{\mu} z = \frac{D_1 h}{\mu} \qquad (5.3.6)$$

Integrating this equation and applying the boundary conditions :

$$\begin{aligned} u_1 &= 0 \quad \text{at} \quad z = 1 \\ &= 0 \quad \text{at} \quad z = 0 \end{aligned} \qquad (5.3.7)$$

we get

$$\frac{u_1}{U} = \frac{\rho h}{2\mu} z(1-z) \qquad (5.3.8)$$

and

$$D_1 = \frac{U\rho}{2}$$

We observe from this equation that u_1 is always positive and symmetric in between the plates, and has the maximum value at $z = 1/2$ i.e. midway between the plates.

Again collecting the coefficients of V^2, we once again obtain a linear differential equation for u_2 as :

$$\frac{\partial u_2}{\partial z} = \frac{D_2 h}{\mu} + \frac{\rho\lambda U}{\mu} - \frac{U\rho^2 h^2}{2\mu^2} z + \frac{U\rho^2 h^2}{2\mu^2} z^2 \qquad (5.3.9)$$

Again integrating this equation under the boundary conditions

$$\begin{aligned} u_2 &= 0 \quad \text{at} \quad z = 1 \\ &= 0 \quad \text{at} \quad z = 0 \end{aligned} \qquad (5.3.10)$$

we get :

$$\frac{u_2}{U} = \frac{\rho^2 h^2}{12\mu^2} z(z-1)(2z-1) \qquad (5.3.11)$$

and $$D_2 = \frac{U\rho^2 h}{12\mu} - \frac{U\rho\lambda}{h}$$

Equation (5.3.11) shows that u_2 is always positive in the upper half region while it is negative in the lower half of the region.

Similarly the differential equation for u_3 obtained by collecting the coefficients of V^3 is

$$\frac{\partial u_3}{\partial z} = \frac{D_3 h}{\mu} + \frac{U\rho^2\lambda}{2\mu^2}(1-2z) - \frac{U\rho^3 h^2}{12\mu^3} z(2z-1)(z-1) \qquad (5.3.12)$$

The integral of which under the boundary condition viz.

$$u_3 = 0 \quad \text{at} \quad z = 1$$
$$\quad = 0 \quad \text{at} \quad z = 0 \qquad (5.3.13)$$

gives

$$\frac{u_3}{U} = \frac{\rho^2 \lambda h}{2\mu^2} z(1-z) - \frac{\rho^3 h^3}{24\mu^3} [z(1-z)]^2 \qquad (5.3.13)$$

and $\quad D_3 = 0$

Thus u_3 is always positive if a new parameter $k = \frac{\rho h^2}{\mu \lambda}$ is such that

$$K < 48 \qquad (5.3.15)$$

Again the coefficients of V^4 give the linear differential equation for u_4 as

$$\frac{\partial u_4}{\partial z} = \frac{D_4 h}{\mu} + \frac{U\rho^4 h^4}{24\mu^4}[z(1-z)]^2 + \frac{U\rho^3 h^2 \lambda}{12\mu^3}(12z^2 - 12z + 1) \qquad (5.3.16)$$

which on integration under the usual boundary conditions

$$u_4 = 0 \quad \text{at} \quad z = 1$$
$$\quad = 0 \quad \text{at} \quad z = 0 \qquad (5.3.17)$$

gives

$$\frac{u_4}{U} = \frac{\rho^4 h^4}{120\mu^4} z(z-1)(6z^3-9z^2+z+1)$$

$$+ \frac{\rho^3 h^2 \lambda}{6\mu^3} z(z-1)(2z-1) \qquad (5.3.18)$$

and $$D_4 = \frac{U\rho^3 h\lambda}{12\mu^4} - \frac{U\rho^4 h^3}{720\mu^3}$$

The coefficients of V^5 give the following differential equation for u_5 as

$$\frac{\partial u_5}{\partial z} = \frac{D_5 h}{\mu} - \frac{U\rho^5 h^5}{720\mu^5} z(z-1)(6z^3-9z^2+z+1)$$

$$- \frac{U\rho^4 h^3 \lambda}{4\mu^4} z(1-2z)(1-z) + \frac{U\rho^3 h\lambda^2}{2\mu^3}(1-2z) \qquad (5.3.19)$$

and this on integration under the usual boundary conditions

$$u_5 = 0 \quad \text{at} \quad z = 1$$

$$= 0 \quad \text{at} \quad z = 0 \qquad (5.3.20)$$

gives

$$\frac{u_5}{U} = -\frac{\rho^5 h^5}{1440\,\mu^5}\, z^2(1-2z)^2(2z^2-2z+1)$$

$$-\frac{\rho^4 h^3 \lambda}{80\,\mu^4}\cdot z^2(1-z)^2 + \frac{\rho^3 h \lambda^2}{2\mu^3}\, z(1-z) \qquad \textbf{(5.3.21)}$$

and $\qquad D_5 = 0$

In this way the values of $u_6, u_7 \ldots$ etc. may be obtained to any desired extent but the algebra becomes quite heavy as we proceed further.

Thus the velocity profile becomes :

$$\frac{u}{U} = z + \frac{VR}{U}\, z(1-z)\Big[1 + \frac{RV}{6U}(1-2z)$$

$$-\frac{V^2R^2}{12U^2}\cdot z(1-z) - \frac{V^3R^3}{360\,U^3}(6z^3-9z^2+z+1)+$$

$$\frac{V^4R^4}{720\,U^4}\, z(z-1)(2z^2-2z+1)\ldots\Big] + \frac{R^2V^3\lambda}{2U^2h}\, z(1-z)$$

$$\Big[1+\frac{VR}{3U}(2z-1) - \frac{R^2V^2}{4U^2}z(1-z)\ldots\Big] - \frac{R^3V^5\lambda^2}{2U^3h^2}\, z(1-z) \qquad \textbf{(5.3.22)}$$

where

$$R = \frac{Uh}{\mu/\rho}$$

5.4 SOLUTION FOR SMALL RELAXATION TIME BUT FOR ALL VALUES OF THE VELOCITY OF SUCTION AND INJECTION

The results (5.3.8) to (5.3.21) show that $u_0, u_1, u_2 \ldots.$ are all functions of λ in the form :

$$u = f(z) + \lambda \phi(z) + \lambda^2 \psi(z) \qquad (5.4.1)$$

To verify the results we choose this value of along with

$$D = D_0 + \lambda D_1 + \lambda^2 D_2 \qquad (5.4.2)$$

and substituting those values in (5.2.6) and then the value of t_{xy} thus obtained in (5.2.0) we get

$$\left[(-\rho v f + D_0) + \lambda(-\rho v \phi + D_1) + \lambda^2(-\rho v \psi + D_2) \right] \times$$

$$\left[h f' + \lambda (h\phi' + v f'') + \lambda^2 (h\psi' + v\phi'') \right] +$$

$$2\rho v^2 (\lambda f'^2 + 2\lambda^2 f'\phi')$$

$$= \mu \left[f'^2 + 2\lambda f'\phi' + \lambda^2 (2f'\psi' + \phi'^2) \right] \qquad (5.4.3)$$

Collecting the terms independent of λ we get the differential equation for $f(z)$ as

$$f' + R f = \frac{D_0 h}{\mu} \qquad (5.4.4)$$

where $R = \frac{Vh}{\mu/\rho}$ and prime denotes differentiation with respect to z .

Integrating equation (5.4.4) and applying the boundary conditions :

$$\begin{aligned} f &= 0 \quad \text{at} \quad z = 0 \\ &= U \quad \text{at} \quad z = 1 \end{aligned} \tag{5.4.5}$$

we get

$$\frac{f}{U} = \frac{1-e^{-Rz}}{1-e^{-R}} \tag{5.4.6}$$

and

$$D_0 = \frac{U R \mu}{h(1-e^{-R})}$$

The expansion of this expression for $f(z)$ in powers of R comes out to be

$$\frac{f(z)}{U} = z + \frac{R}{2} z(1-z) + \frac{R^2}{12} z(z-1)(2z-1)$$

$$- \frac{R^3}{24} z^2(1-z)^2 - \frac{R^4}{720} z(1-z)(6z^3-9z^2+z+1) \tag{5.4.7}$$

The differential equation for $\phi(z)$ is obtained by collecting the coefficients of λ , which again comes out to be the linear differential equation

$$\phi'(z) + R\phi(z) = \frac{D_1 h}{\mu} + \frac{UVR^2 e^{-Rz}}{h(1-e^{-R})} \qquad (5.4.8)$$

The solution of this equation under the boundary conditions

$$\phi(z) = 0 \quad \text{at} \quad z = 0$$
$$= 0 \quad \text{at} \quad z = 1 \qquad (5.4.9)$$

is found to be

$$\frac{\phi(z)}{U} = \frac{VR^2 e^{-R}}{h(1-e^{-R})^2}\left[z(e^{R}-1)e^{-Rz} + e^{-Rz} - 1\right]$$

and (5.4.10)

$$D_1 = -\frac{R^3 UV e^{-R}}{h^2(1-e^{-R})^2} \cdot \mu$$

Similarly collecting the coefficients of λ^2 we get a linear differential equation for $\psi(z)$ as :

$$\psi'(z) + R\psi(z) = \frac{D_2 h}{\mu} + \frac{UV^2R^3e^{-Rz}}{h^2(1-e^{-R})^2}\left[(1-e^{-R})(1-Rz) - Re^{-R}\right] \qquad (5.4.11)$$

and its solution under the usual boundary conditions is found to be

$$\frac{\psi(z)}{U} = \frac{v^2 R^3}{h^2(1-e^{-R})^3}\Big[e^{-R}(1-e^{-Rz})\{Re^{-R}-(1-e^{-R})(1-\tfrac{R}{2})\}$$

$$+ e^{-Rz}(1-e^{-R})\{(z-\tfrac{R}{2}z^2)(1-e^{-R})-Rze^{-R}\}\Big] \qquad \textbf{(5.4.12)}$$

and $$D_2 = \frac{U v^2 R^4 \mu e^{-R}}{h^2(1-e^{-R})^3}\Big[Re^{-R}-(1-e^{-R})(1-\tfrac{R}{2})\Big]$$

5.5 DISCUSSIONS OF THE RESULTS

The equation (5.4.1) can now be written as :

$$\frac{u}{U} = \frac{1-e^{-Rz}}{1-e^{-R}} + A\frac{R^2 e^{-R}}{(1-e^{-R})^2}\Big[z(e^{R}-1)e^{-Rz}+e^{-Rz}-1\Big]$$

$$+ A^2\frac{R^3}{(1-e^{-R})^3}\Big[e^{-R}(1-e^{-Rz})\{Re^{-R}-(1-e^{-R})(1-\tfrac{R}{2})\}$$

$$+ e^{-Rz}(1-e^{-R})\{(z-\tfrac{R}{2}z^2)(1-e^{-R})-Rze^{-R}\}\Big] \qquad \textbf{(5.5.1)}$$

where $$A = \frac{\lambda v}{h}$$

We have tabulated the values of $f(z)$, $\phi(z)$ and $\psi(z)$ in the whole region between two plates and observe that if $R=0$ the velocity is the same as for the elastic and inelastic liquids (ii) for $A = \frac{\lambda v}{h} = \cdot 1$ we see that the existing

velocity profile becomes more bulging forward for the values of $R = .5$ due to the superposition of the terms $\phi(z)$ and $\psi(z)$. For increasing R the primary velocity field $f(z)$ attains the velocity of the upper moving plate more rapidly than for smaller R. The contribution due to $\phi(z)$ goes on increasing with the increase in R but for values of $R > 4$ this contribution starts decreasing. Similarly the contribution of $\psi(z)$ term increases till $R = 4$ but starts decreasing when R attains larger values than 4.

Table 5.1 : The non-dimensional velocity field $\frac{u}{U} = \frac{f(z)}{U}$

for inelastic liquid. (i.e. $A = \frac{\lambda V}{h} = 0$ **)**

for various values of Reynolds number R **.**

z \ R	.5	1	2	4	8
00	.000000	.000000	.00000	.00000	.00000
0.1	.12395	.150545	.20964	.33583	.55086
0.2	.24185	.28676	.38128	.56095	.79237
0.3	.35401	.41002	.52181	.71184	.90959
0.4	.46009	.52165	.63686	.81299	.95956
0.5	.56918	.62246	.73106	.88080	.98201
0.6	.658709	.71377	.80818	.92625	.99210
0.7	.76063	.79639	.87132	.95671	.99664
0.8	.83788	.87115	.93302	.97713	.99367
0.9	.92097	.93279	.96536	.99022	.99959
1.0	1.00000	1.00000	1.00000	1.00000	1.00000

Table 5.2 : The Superposition of the coefficient of λ (i.e. $\phi(z)/U$) over the velocity field given by $f(z)/U$ for various values of Reynolds number R and $A = \frac{\lambda v}{h} = \cdot 1$

z \ R	.5	1.0	2.0	4.0	8.0
00	.000000	.000000	.000000	.000000	.000000
0.1	.001267	.00555	.024750	.099227	.282494
0.2	.002178	.009220	.038148	.129723	.266800
0.3	.002764	.01130	.043496	.126021	.172283
0.4	.003054	.01206	.043273	.107365	.102325
0.5	.0030767	.01175	.039322	.083996	.068521
0.6	.002867	.01055	.033003	.061064	.029482
0.7	.002418	.00864	.025303	.040818	.014432
0.8	.001783	.00617	.016931	.023980	.006365
0.9	.000970	.00325	.008384	.010502	.002155
1.0	.000000	.00000	.000000	.000000	.000000

Table 5.3 : The Superposition of the coefficient of λ^2 (i.e. $\psi(z)/U$) over the combined velocity field given by $\frac{u}{U} = \{f(z) + \lambda\phi(z)\}/U$ for various values of Reynolds number R and $A = \frac{\lambda\nu}{h} = \cdot 1$.

z \ R	.5	1.0	2.0	4.0	8.0
00	.000000	.000000	.000000	.000000	.000000
0.1	.000667	.000599	.005268	.036009	.140304
0.2	.001127	.000961	.007535	.037978	.044026
0.3	.001407	.001138	.007940	.028201	-.023559
0.4	.001529	.001174	.007268	.017093	-.045379
0.5	.001514	.001103	.006047	.008010	-.041963
0.6	.001382	.000965	.004622	.002140	-.030356
0.7	.001150	.000754	.003207	-.000895	-.018757
0.8	.000833	.000518	.001929	-.001804	-.009344
0.9	.000446	.000263	.000852	-.001308	-.003800
1.0	.000000	.000000	.000000	.000000	.000000

PART II

FLOW IN AN ANNULUS

5.6 BASIC EQUATIONS

We choose the cylindrical polar coordinates (r, θ, z) with the z-axis along the axis of the cylinder. In the steady state of the liquid flow the velocity pattern is given by

$$u = u(r) \quad , \quad v = v(\theta) = 0 \quad , \omega = \omega(r) \qquad (5.6.1)$$

If we consider the flow at a sufficient distance from the end so that u, v and ω may be taken to be independent of z and then the stress, strain-rate equations become

$$t_{rr} + \lambda \left[v \frac{\partial t_{rr}}{\partial r} - 2 \frac{\partial v}{\partial r} \cdot t_{rr} \right] = 2\mu \frac{\partial v}{\partial r}$$

$$t_{rz} + \lambda \left[v \frac{\partial t_{rz}}{\partial r} - \frac{\partial u}{\partial r} \cdot t_{rr} - \frac{\partial v}{\partial r} t_{rz} \right] = \mu \frac{\partial u}{\partial r}$$

$$(5.6.2)$$

$$t_{zz} + \lambda \left[v \frac{\partial t_{zz}}{\partial r} - 2 \frac{\partial u}{\partial r} \cdot t_{rz} \right] = 0$$

$$t_{\theta\theta} + \lambda \left[v \frac{\partial t_{\theta\theta}}{\partial r} - 2 \frac{v}{r} \cdot t_{\theta\theta} \right] = 2\mu \frac{v}{r}$$

$$t_{r\theta} = t_{\theta z} = 0$$

The equation of continuity is

$$\frac{\partial v}{\partial r} + \frac{v}{r} = 0 \tag{5.6.3}$$

The equation of momentum is

$$\rho v \frac{\partial u}{\partial r} = \frac{\partial t_{rz}}{\partial r} + \frac{1}{r} t_{rz} \tag{5.6.4}$$

The boundary conditions are

$$\begin{aligned} v &= v_a \quad \text{at} \quad r = a \\ &= v_b \quad \text{at} \quad r = b \end{aligned} \tag{5.6.5}$$

where a and b are the radii of outer and inner cylinders respectively.

The inner cylinder is moving parallel to its own axis with velocity U such that

$$\begin{aligned} u &= 0 \quad \text{at} \quad r = a \\ &= U \quad \text{at} \quad r = b \end{aligned} \tag{5.6.7}$$

The equation of continuity on integration gives

$$\frac{a v_a}{r} = \frac{b v_b}{r} = v \tag{5.6.8}$$

The equation (5.6.4) on integration gives

$$t_{rz} = \frac{\rho a v_a}{r} . u + \frac{D}{r} \tag{5.6.9}$$

where D is the constant of integration.

Eliminating t_{rr} from the first and second equation of (5.6.2), we get

$$t_{rz} \frac{\partial u}{\partial r} + \lambda \left[- \frac{a v_a}{r} . t_{rz} . \frac{\partial^2 u}{\partial r^2} + \frac{2 a v_a}{r} \frac{\partial u}{\partial r} . \frac{\partial t_{rz}}{\partial r} + \frac{3 a v_a}{r^2} . \frac{\partial u}{\partial r} . t_{rz} \right] = \mu \left(\frac{\partial u}{\partial r} \right)^2 \tag{5.6.10}$$

Substituting the value of t_{rz} from (5.6.9) in (5.6.10), we get the differential equation in the form

$$\frac{\partial u}{\partial r} \left[\frac{\rho a v_a}{r} u + \frac{D}{r} + \frac{2 \lambda a^2 v_a^2}{r^2} . \frac{\partial u}{\partial r} + \frac{\lambda \rho a^2 v_a^2}{r^3} u + \frac{\lambda a D v_a}{r^3} \right]$$

$$- \frac{\partial^2 u}{\partial r^2} \left[\frac{\lambda \rho a^2 v_a^2}{r^2} . u + \frac{D \lambda a v_a}{r^2} \right] = \mu \left(\frac{\partial u}{\partial r} \right)^2 \tag{5.6.11}$$

Transforming this into non-dimensional form by substitutions

$$u' = \frac{u}{v_a} \quad ; \quad \lambda' = \frac{\lambda v_a}{a}$$

$$D' = \frac{D}{\rho a v_a} \quad ; \quad \xi = \frac{r}{a} \tag{5.6.12}$$

$$\mu' = \frac{\mu}{\rho a v_a} = \frac{1}{R} \quad ; \quad U' = \frac{U}{v_a}$$

we get

$$\frac{\partial u'}{\partial \xi}\left[\frac{u'}{\xi}+\frac{D'}{\xi}+\frac{\lambda' u'}{\xi^3}+\frac{\lambda' D'}{\xi^3}+\frac{2\lambda'}{\xi^2}\cdot\frac{\partial u'}{\partial \xi}\right]$$

$$-\lambda' \frac{\partial^2 u'}{\partial \xi^2}\left(\frac{u'}{\xi}+\frac{D'}{\xi^2}\right) = \mu'\left(\frac{\partial u'}{\partial \xi}\right)^2 \qquad (5.6.13)$$

This differential equation is non-linear and of the second order and the complete analytical solution is not possible. We therefore, proceed to solve (5.6.13) by the series method as before.

5.7 SERIES SOLUTION FOR SMALL RELAXATION TIME BUT FOR ALL VELOCITIES OF SUCTION AND INJECTION

Let us assume the solution of (5.6.13) of the form

$$u' = f(\xi) + \lambda\, \phi(\xi) + \lambda'^2 \psi(\xi) \qquad (5.7.1)$$

together with $D' = D_0' + \lambda' D_1' + \lambda'^2 D_2'$

Collecting the terms independent of λ' from (5.6.13) after substituting (5.7.1) for the inelastic case, we get

$$\frac{\partial f}{\partial \xi} - f\cdot\frac{R}{\xi} = D_0' \frac{R}{\xi} \qquad (5.7.2)$$

The solution of this equation under the boundary conditions

$$f = 0 \quad \text{at} \quad \xi = 1$$

$$= U' \quad \text{at} \quad \xi = \sigma = \frac{b}{a} \qquad (5.7.3)$$

is

$$f(\xi) = U' \frac{(\xi^R - 1)}{(\sigma^R - 1)} \qquad (5.7.4)$$

and

$$D_0' = \frac{U'}{(\sigma^R - 1)}$$

Similarly collecting the coefficients of λ^1 from the equation (5.6.13) the differential equation for ϕ is found to be of the form

$$\frac{\partial \phi}{\partial \xi} - \frac{R}{\xi} \cdot \phi = R \frac{D_1'}{\xi} + \frac{U' R (R+2) \xi^R}{(\sigma^R - 1) \xi^3} \qquad (5.7.5)$$

whose integral under the boundary conditions

$$\phi = 0 \quad \text{at} \quad \xi = 1$$

$$= 0 \quad \text{at} \quad \xi = \sigma \qquad (5.7.6)$$

is

$$\frac{\phi}{U'} = \frac{R(R+2)}{2(\sigma^R - 1)^2} \left[-\xi^{R-2} \cdot \sigma^R + \xi^{R-2} + \sigma^R - \sigma^{R-2} \xi^R - \xi^R \right]$$

and

$$D_1' = \frac{U' R (R+2)}{2(\sigma^R - 1)} \left[\frac{\sigma^{R-2} - \sigma^R}{\sigma^R - 1} \right] \qquad (5.7.7)$$

Again collecting the coefficients of λ'^2 from (5.6.13), after substituting from (5.7.1), the differential equation for ψ is found to be of the form :

$$\frac{\partial \psi}{\partial \xi} - \frac{R}{\xi}\cdot\psi = \frac{R D_2'}{\xi} + B(R-2)(R+2)(1-\sigma^R)\xi^{R-5}$$

$$- B\cdot R\cdot(R+2)(1-\sigma^{R-2})\xi^3 \qquad (5.7.8)$$

where $$B = \frac{U' R (R+2)}{2(\sigma^R-1)^2}$$

The solution of which under the boundary conditions :

$$\psi = 0 \quad \text{at} \quad \xi = 1$$

$$= 0 \quad \text{at} \quad \xi = \sigma \qquad (5.7.9)$$

is $$\frac{\psi}{U'} = \frac{R(R+2)^2}{8(\sigma^R-1)^2}\Big[(R-2)(\sigma^R-1) \times$$

$$\left\{(\sigma^{R-4}-\sigma^R) - (\sigma^{R-4}-1)\xi^R + (\sigma^R-1)\xi^{R-4}\right\}$$

$$+ 2R(\sigma^{R-2}-1)\left\{(\sigma^R-\sigma^{R-2}) + (\sigma^{R-2}-1)\xi^R\right.$$

$$\left.- (\sigma^R-1)\xi^{R-2}\right\}\Big] \qquad (5.7.10)$$

and

$$D_2' = \frac{U'(R+2)^2}{8(\sigma^R - 1)^2}\Big[(R-2)(\sigma^R - 1)(\sigma^R - \sigma^{R-4})$$

$$+ 2R(\sigma^{R-2} - 1)(\sigma^{R-2} - \sigma^R)\Big]$$

5.8 SERIES SOLUTION FOR SMALL VELOCITIES OF SUCTION AND INJECTION BUT FOR ALL RELAXATION TIME

We assume the solution of (5.6.11) of the form

$$u = u_0 + \vartheta_a u_1 + \vartheta_a^2 u_2 \dots$$

and (5.8.1)

$$D = D_0 + \vartheta_a D_1 + \vartheta_a^2 D_2 \dots$$

Substituting these values in (5.6.12) and collecting the terms independent of ϑ_a i.e. the case of no suction or injection, the differential equation for u_0 is

$$\frac{\partial u_0}{\partial r} = \frac{D_0}{\mu} \cdot \frac{1}{r} \tag{5.8.2}$$

Integrating this equation under the boundary conditions

$$u_0 = 0 \quad \text{at} \quad r = a$$

(5.8.3)

$$= U \quad \text{at} \quad r = b$$

we get

$$\frac{u_0}{U} = \frac{\ln \xi}{\ln a}$$

and (5.8.4)

$$D_0 = \frac{U\mu}{\ln \sigma}$$

where $\xi = \frac{r}{a}$ and $\sigma = \frac{b}{a}$

This agrees with the result for ordinary visco-elastic fluid for the case of no suction and injection.

Collecting the coefficients of ν_a from (5.6.12) after substitutions from (5.8.1) we again get a linear differential equation for u_1 :

$$\frac{\partial u_1}{\partial r} = \frac{D_0}{\mu} \frac{1}{r} + \frac{Ua\rho}{\mu \ln \sigma} \cdot \frac{\ln \xi}{r} + \frac{2Ua\lambda}{\ln \sigma} \cdot \frac{1}{r^3} \qquad (5.8.5)$$

Integrating this equation under the boundary conditions

$$u_1 = 0 \quad \text{at} \quad r = a$$

(5.8.6)

$$= 0 \quad \text{at} \quad r = b$$

we get

$$\frac{u_1}{U} = \frac{a\lambda}{\ln\sigma}\left[\frac{(a^2-b^2)}{a^2b^2}\frac{\ln\varsigma}{\ln\sigma} - \left(\frac{1}{r^2}-\frac{1}{a^2}\right)\right.$$

$$\left. + \frac{a\rho\ln\varsigma}{2\mu}\cdot\frac{\ln r/b}{\ln\sigma}\right] \qquad (5.8.7)$$

and

$$D_1 = \frac{Ua\lambda}{(\ln\sigma)^2}\cdot\frac{(a^2-b^2)}{a^2b^2} - \frac{Ua\rho}{2\mu}$$

Similarly collecting the coefficients of v_a^2 from (5.6.12) after substitutions from (5.8.1) we get

$$\frac{\partial u_2}{\partial r} = \frac{UD_2}{\mu}\frac{1}{r} + \frac{Ua^2\lambda\rho}{\mu\ln\sigma}\cdot\frac{1}{r}\cdot\left[\frac{a^2-b^2}{a^2b^2}\frac{\ln\varsigma}{\ln\sigma} + \frac{\ln\left(\frac{r^2}{ab}\right)}{r^2} + \frac{1}{a^2}\right]$$

$$- \frac{2Ua^2\lambda^2}{r\ln\sigma}\left[\frac{1}{r^4} + \frac{a^2-b^2}{a^2b^2}\cdot\frac{1}{r^2}\cdot\frac{1}{\ln\sigma}\right]$$

$$- \frac{Ua^2\rho^2}{2\mu^2 r}\frac{\ln(r/b)}{\ln\sigma} \qquad (5.8.8)$$

and this can also be integrated under the same boundary conditions viz

$$u_2 = 0 \quad \textbf{at} \quad r = a$$

$$= 0 \quad \textbf{at} \quad r = b$$

5.9 DISCUSSION OF THE RESULTS

The solution obtained for small relaxation time parameter λ' can be written as

$$\frac{u}{U'} = \frac{(\xi^{R}-1)}{(\sigma^{R}-1)} + \frac{R(R+2)\lambda'}{2(\sigma^{R}-1)^{2}}\Big[-\xi^{R-2}\sigma^{R} + \sigma^{R} + \xi^{R-2} - \sigma^{R-2} + \sigma^{R-2}\cdot\xi^{R} - \xi^{R}\Big]$$

$$+ \frac{R(R+2)^{2}\lambda'^{2}}{8(\sigma^{R}-1)^{2}}\Big[(R-2)(\sigma^{R}-1)\Big\{(\sigma^{R-4}-\sigma^{R}) - (\sigma^{R-4}-1)\xi^{R} + (\sigma^{R}-1)\xi^{R-4}\Big\}$$

$$+ 2R(\sigma^{R-2}-1)\Big\{(\sigma^{R}-\sigma^{R-2}) + (\sigma^{R-2}-1)\xi^{R} - (\sigma^{R}-1)\xi^{R-2}\Big\}\Big] \quad (5.9.1)$$

where $R = \dfrac{a\,\nu_a}{\mu/\rho}$ and $\lambda' = \dfrac{\lambda\,\nu_a}{a}$

In the following tables we have calculated functions $f(\xi)$, $\phi(\xi)$ and $\psi(\xi)$ respectively. From these we observe that (i) if $R=0$, the velocity is the same as for elastic and inelastic liquid (ii) if $0<R<2$ velocity decreases as compared with inelastic liquid. (iii) if $R=2$ the velocity remains unchanged (iv) if $R>2$ the velocity increases as compared with inelastic liquid.

Table 5.4 : The non-dimensional velocity field $\frac{u}{U'} = f(\zeta)/U'$ for inelastic liquid (i.e. $\lambda' = \frac{\lambda \nu_a}{a} = 0$) for various values of Reynolds number R.

	ζ \ R	.5	1.0	2.0	4.0	8.0
	0.2	1.000000	1.000000	1.000000	1.000000	1.000000
	0.3	.818177	.875000	.947916	.992483	.999736
	0.4	.664594	.750000	.875000	.975261	.996347
	0.5	.529848	.625000	.781250	.939002	.996056
σ =.2	0.6	.407758	.500000	.666665	.871794	.983206
	0.7	.295484	.375000	.531250	.761117	.947354
	0.8	.190293	.25000	.375000	.591346	.852229
	0.9	.092832	.125000	.197916	.344451	.569584
	1.0	.000000	.000000	.000000	.000000	.000000
	0.5	1.000000	1.000000	1.000000	1.000000	1.000000
	0.6	.769575	.800000	.858333	.928426	.987059
	0.7	.557677	.600000	.680000	.810560	.947069
σ =.5	0.8	.360448	.400000	.480000	.629760	.835491
	0.9	.175206	.200000	.258233	.366926	.571766
	1.0	.000000	.000000	.000000	.000000	.000000

Table 5.5 : The superposed velocity $\phi(\rho)/U'$ over the non-dimensional velocity field $\frac{u}{U'} = \frac{f(\rho)}{U'}$ for elastic liquid $\lambda' = \frac{\lambda v_a}{a} = .1$ and various values of Reynolds number R.

	ρ \ R	.5	1.0	2.0	4.0	8.0
	0.2	.000000	.000000	.000000	.000000	.000000
	0.3	-.366724	-.218750	.000000	.052524	.02408
	0.4	-.431461	-.281250	.000000	.116494	.01252
	0.5	-.403141	-.281250	.000000	.182022	.04663
σ =.2	0.6	-.339130	-.250000	.000000	.236686	.11920
	0.7	-.260121	-.200233	.000000	.265232	.23978
	0.8	-.174879	-.140525	.000000	.249630	.37728
	0.9	-.087496	-.072917	.000000	.169078	.403756
	1.0	.000000	.000000	.000000	.000000	.000000
	0.5	.000000	.000000	.000000	.000000	.000000
	0.6	-.05511	-.040000	.000000	.072020	.073458
	0.7	-.066621	-.051428	.000000	.125337	.196425
σ =.5	0.8	-.065205	-.045000	.000000	.143769	.339450
	0.9	-.031824	-.026667	.000000	.105963	.378572
	1.0	.000000	.000000	.000000	.000000	.000000

Table 5.6 : The superposed velocity $\psi(\eta)/U'$ over the combined non-dimensional velocity field $\frac{u}{U'} = \{f(\eta) + \lambda\phi(\eta)\}/U'$ for the elastic liquid $\lambda' = \frac{\lambda \nu_a}{a} = \cdot 1$ and various values of Reynolds number R .

	η \ R	.5	1.0	2.0	4.0	8.0
	0.2	.000000	.000000	.000000	.000000	.000000
	0.3	.711484	.683958	.000000	-.060577	.028992
	0.4	.671613	.712969	.000000	-.134201	.086025
σ =.2	0.5	.547714	.624375	.000000	-.209609	.155506
	0.6	.420475	.506657	.000000	-.272663	.190638
	0.7	.302131	.381040	.000000	-.305647	.126649
	0.8	.193457	.263652	.000000	-.287574	-.056048
	0.9	.093209	.126443	.000000	-.194778	-.256584
	1.0	.000000	.000000	.000000	.000000	.000000
	0.5	.000000	.000000	.000000	.000000	.000000
	0.6	.011116	.016167	.000000	-.061881	.040569
	0.7	.011530	.017967	.000000	-.112804	-.011207
σ =.5	0.8	.008470	.01027	.000000	-.129393	-.180134
	0.9	.004338	.007563	.000000	-.093058	-.33784
	1.0	.000000	.000000	.000000	.000000	.000000

CHAPTER VI

FLOW OF A SPECIAL TYPE OF NON-NEWTONIAN FLUID BETWEEN TWO PARALLEL PLATES AND THROUGH A CIRCULAR PIPE

6.1 INTRODUCTION

In this chapter we have obtained the exact analytical solutions for the flow of the liquid characterised by the rheological equation (1.3.2) between two parallel plates and in a circular pipe. The exact analytical solutions are, however, very complicated in the general case and therefore we have discussed only some special cases in greater details. We have also discussed the series solutions for fluids for which either the Newtonian or the power-law term can be regarded as a perturbation over the other more dominant term. The two parts of this chapter deal with two separate cases viz.

(i) flow of this fluid between two parallel plates

(ii) flow of the same fluid in a circular pipe.

6.2 BASIC EQUATIONS

Let us consider the fluid with constitutive equation (1.2.2) flowing between two parallel plates, a distance 'h' apart under a constant pressure gradient. Using the Cartesian coordinates for the equations of motion when the velocity components are of the form :

$$u = u(y) \quad , \quad v = 0 \quad \text{and} \quad w = 0 \qquad (6.2.1)$$

The components of the strain-rate are given by

$$e_{xx} = e_{yy} = 0 \quad ; \quad e_{xy} = \frac{\partial u}{\partial y} \qquad (6.2.2)$$

and the stress components are given by

$$t_{xx} = t_{yy} = 0$$

$$t_{xy} = \mu\left(\frac{\partial u}{\partial y}\right) + \frac{\mu}{\alpha}\left|\frac{\partial u}{\partial y}\right|^{n-1} \cdot \frac{\partial u}{\partial y} \qquad (6.2.3)$$

The equations of motion for Cartesian coordinates with constant pressure gradient are :

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \frac{\partial t_{xx}}{\partial x} + \frac{\partial t_{xy}}{\partial y}$$

$$(6.2.4)$$

$$\rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \frac{\partial t_{xy}}{\partial x} + \frac{\partial t_{yy}}{\partial y}$$

The second equations of motion (6.2.4) gives $\partial p/\partial y = 0$ which shows that there is no pressure gradient in the y-direction and the pressure profile is independent of y. Here $\partial p/\partial x$ is the pressure gradient that exists along the length of the channel, and the first equation of motion gives :

$$\frac{\partial p}{\partial x} = \frac{\partial}{\partial y}\left[\mu \frac{\partial u}{\partial y} + \frac{\mu}{a}\left|\frac{\partial u}{\partial y}\right|^{n-1} \cdot \frac{\partial u}{\partial y}\right] \qquad (6.2.5)$$

which on integration gives

$$\mu \frac{\partial u}{\partial y} + \frac{\mu}{a}\left|\frac{\partial u}{\partial y}\right|^{n-1} \cdot \frac{\partial u}{\partial y} = \frac{dp}{dx} \cdot y + A \qquad (6.2.6)$$

Since $\partial u/\partial y$ is essentially negative throughout the channel we write (6.2.6) as

$$\mu\left(-\frac{\partial u}{\partial y}\right) + \frac{\mu}{a}\left|\frac{\partial u}{\partial y}\right|^{n-1} \cdot \left(-\frac{\partial u}{\partial y}\right) = -\left[\frac{dp}{dx} \cdot y + A\right] \qquad (6.2.7)$$

The boundary conditions are

$$\begin{aligned} u &= 0 \quad \text{at} \quad y = h \\ \frac{\partial u}{\partial y} &= 0 \quad \text{at} \quad y = 0 \end{aligned} \qquad (6.2.8)$$

6.3 EXACT SOLUTION

Let us put $\frac{\partial u}{\partial y} = u'$ and $-\frac{dp}{dx} = P$ such that (6.2.7) is written as

$$-A + P.y = \mu(-u') + \frac{\mu}{a}(-u')^{n} \qquad (6.3.1)$$

Differentiating this with respect to u we get

$$\frac{P}{u'} = \left[\mu + n \frac{\mu}{a} (-u')^{n-1} \right] \left(- \frac{\partial u'}{\partial u} \right) \qquad (6.3.2)$$

which on integration gives

$$P.u = -\mu \frac{u'^2}{2} - \frac{n}{n+1} \cdot \frac{\mu}{a} \cdot (-u')^{n+1} + c \qquad (6.3.3)$$

Also we have (6.3.1) written as, since A being zero

$$P.y = \mu(-u') + \frac{\mu}{a} (-u')^{n} \qquad (6.3.4)$$

Multiplying (6.3.4) by $\frac{n}{n+1} (-u')$ and adding to (6.3.3)

we get

$$\mu(n-1) u'^2 + 2n \cdot P. u' \cdot y - 2P. u. (n+1) + 2(n+1) c = 0 \qquad (6.3.5)$$

In principle, this equation along with the original equation

$$\mu u' - \frac{\mu}{a} (-u')^{n} + Py = 0$$

can be solved for u' (i.e. the velocity gradient) for any n. The constant of integration c being determined from the eliminant of u'. But this becomes too much algebraically complicated and as such we give here the solutions for a few convenient values of n viz. $n = 1/2$, 1, and 2 as this is the range in which generally n lies for the power-law fluids.

Case 1 : n = 1/2

The two equations (6.3.5) and (6.3.4) reduce to the following for the particular value of n = 1/2

$$\mu u'^2 - 2P.u'.y + 6(c - Pu) = 0$$

$$\mu u'^2 + u'(\mu_a^2 + 2\mu.P.y) + P^2 y^2 = 0 \qquad (6.3.6)$$

The u' eliminant gives

$$36\,c^2.\mu^3 + c\,(-72.P.u.\mu^3 + 36.P^2.y^2.\mu^2 + 6\mu_a^4 + 36.P.y.\mu.\mu_a^2) +$$

$$(36P.u^2\mu^3 + 9P^4\mu.y^4 - 36P^3\mu^2y^2u + 2P^3y^3\mu_a^3 - 12P^2u.y\mu\mu_a^2) = 0 \qquad (6.3.7)$$

Applying the boundary condition that $u = 0$ at $y = h$, we get another equation for c as :

$$36\,c^2\mu^3 + 6c\,(6P^2h^2\mu^2 + \mu_a^4 + 6.P.\mu.\mu_a^2.h)$$

$$+ (9P^4.h^4.\mu + 2P^3h^3\mu_a^2) = 0 \qquad (6.3.8)$$

Again eliminating c from these two equations we get a biquadratic equation which on solution will give the velocity profile.

Case 2 : n = 1

For n = 1, the problem reduces to the Newtonian case where the coefficient of viscosity is $(\mu+\mu_a)$ instead of μ .

Case 3 : n = 2

For n = 2, the equations (6.3.5) and (6.3.4) reduce to

$$\mu u'^2 + 4.P.y.u' + 6(c-Pu) = 0 \qquad (6.3.9)$$

and $$\mu_a u'^2 - \mu u' + Py = 0$$

The u' eliminant is given by

$$(36Pu^2\mu_a^2 + 36P^2.y.u.\mu.\mu_a + 6Pu\mu^3 + 3P^2y^2\mu^2 + 16P^3.y^3.\mu_a)$$

$$-36c^2.\mu_a^2 - 6c(\mu^3 + 6\mu\mu_a Py - 12\mu_a^2 Pu) = 0 \qquad (6.3.10)$$

and this equation under the boundary condition (6.3.8) gives

$$36c^2\mu_a^2 + 6c(\mu^3 + 6\mu\mu_a.Ph) - P^2h^2(3\mu^2 + 16Ph\mu_a) = 0 \qquad (6.3.11)$$

Again eliminating c between these two equations we get a biquadratic equation in u which on solution gives the velocity profile for the particular value of n = 2. We could

try more cases for different values of n but because of the algebraic complications, we give herewith an approximate but convenient method of solution of (6.3.4) by the method of series.

6.4 SERIES SOLUTION WHEN μ_a IS SMALL

The original equation

$$\mu\left(-\frac{\partial u}{\partial y}\right) + \mu_a\left(-\frac{\partial u}{\partial y}\right)^n = -\left(\frac{dp}{dx}y + A\right)$$

under the boundary conditions (6.2.8) becomes

$$-\frac{dp}{dx}\cdot y = \mu\left(-\frac{\partial u}{\partial y}\right) + \mu_a\left(-\frac{\partial u}{\partial y}\right)^n \qquad (6.4.1)$$

Assuming the series solution of the form

$$-\frac{\partial u}{\partial y} = \sum_{m=0}^{\infty} \mu_a^m \cdot x_m \qquad (6.4.2)$$

and substituting the value of $(-\partial u/\partial y)$ in (6.4.1) and collecting the coefficients of various powers of μ_a, we get the relations :

$$\mu x_0 = Py \quad ; \quad \mu x_1 + x_0^n = 0$$

$$\mu x_2 + n\, x_0^{n-1}\cdot x_1 = 0$$

$$\mu x_3 + n\, x_0^{n-1}\cdot x_2 + \frac{n(n-1)}{\underline{|3}}\cdot x_0^{n-2}\cdot x_1^2 = 0$$

$$\mu x_4 + n x_0^{n-1} . x_3 + \frac{n(n-1)}{\underline{|2}} . x_0^{n-2} . (2x_1 x_2) + {}^nC_3 . x_0^{n-3} . x_1^3 = 0$$

$$\mu x_5 + n x_0^{n-1} . x_4 + {}^nC_2 . x_0^{n-2} . (x_2^2 + 2x_1 x_3) \qquad (6.4.3)$$

$$+ {}^nC_3 \; x_0^{n-3} . (3x_1^2 x_3) + {}^nC_1 \; x_0^{n-4} . x_1^4 = 0$$

which on simplifications give

$$x_0 = \left(\frac{Py}{\mu}\right)$$

$$x_1 = - \frac{(Py)^n}{\mu^{n+1}}$$

$$x_2 = \frac{n\,(Py)^{2n-1}}{\mu^{2n+1}}$$

$$x_3 = - \frac{n(3n-1)}{2} \frac{(Py)^{3n-2}}{\mu^{3n+1}} \qquad (6.4.4)$$

$$x_4 = \frac{n(2n-1)(4n-1)}{3} \frac{(Py)^{4n-3}}{\mu^{4n+1}}$$

$$x_5 = - \frac{n(125n^3 - 150n^2 + 55n - 6)}{24} \frac{(Py)^{5n-4}}{\mu^{5n+1}}$$

These give $(-\partial u/\partial y)$ **in ascending powers of** μ_a **as**

$$-\frac{\partial u}{\partial y} = \frac{(Py)}{\mu} - \frac{(Py)^{n}}{\mu^{n+1}}\cdot\mu_a + n\,\frac{(Py)^{2n-1}}{\mu^{2n+1}}\cdot\mu_a^{2}$$

$$-\frac{n(3n-1)}{2}\,\frac{(Py)^{3n-2}}{\mu^{3n+1}} + \frac{n(2n-1)(4n-1)}{3}\,\frac{(Py)^{4n-3}}{\mu^{4n+1}}\cdot\mu_a^{4}$$

$$-\frac{n(125n^{3}-150n^{2}+55n-6)}{24}\,\frac{(Py)^{5n-4}}{\mu^{5n+1}}\,\mu_a^{5}\;\cdots\cdots \qquad (6.4.5)$$

and this on integration under the boundary condition that $u = 0$ at $y = h$ transforms into

$$u = \frac{P}{2\mu}(h^{2}-y^{2}) - \frac{\mu_a . P^{n}}{(n+1)\,\mu^{n+1}}\cdot\left(h^{n+1}-y^{n+1}\right) + \frac{\mu_a^{2}\,P^{2n+1}}{2\,\mu^{2n+1}}\left(h^{2n}-y^{2n}\right)$$

$$-\frac{n\,P^{3n-2}}{2\,\mu^{3n+1}}\,\mu_a^{3}\left(h^{3n-1}-y^{3n-1}\right) + \frac{(4n^{2}-n)}{6}\,\frac{P^{4n-3}\,\mu_a^{4}}{\mu^{4n+1}}\left(h^{4n-2}-y^{4n-2}\right)$$

$$-\frac{(25n^{3}-15n^{2}+2n)}{24}\,\frac{P^{5n-4}\,\mu_a^{5}}{\mu^{5n+1}}\cdot\left(h^{5n-3}-y^{5n-3}\right)\;\cdots\cdots \qquad (6.4.6)$$

Transforming the above equation (6.4.6) into non-dimensional form by substitutions

$$\begin{aligned} x &= h\xi & \qquad P &= \frac{\rho U^{2}}{h}\frac{\partial\bar{p}}{\partial\xi} = \frac{\rho U^{2}\bar{P}}{h} \\ y &= h\eta & R &= \frac{Uh}{\mu/\rho} \qquad\qquad (6.4.7) \\ u &= U\bar{u} & & \\ p &= \rho U^{2}\bar{p} & R_1 &= \frac{U^{n-1}}{h^{n-1}}\cdot\frac{\mu_a}{\mu} \end{aligned}$$

we get

$$\bar{u} = \frac{R}{2}(1-\eta^2) - \frac{R^n . R_1}{(n+1)}(1-\eta^{n+1}) + \frac{R^{2n-1} . R_1^2}{2}(1-\eta^{2n})$$

$$- \frac{n}{2} R^{3n-2} . R_1^3 (1-\eta^{3n-1}) + \frac{(4n^2-n)}{6} R^{4n-3} R_1^4 (1-\eta^{4n-2})$$

$$- \frac{(25n^3-15n^2+2n)}{24} R^{5n-4} . R_1^5 (1-\eta^{5n-3}) \ldots \quad (6.4.8)$$

Again writing $R^{n-1} . R_1 = k$, we have

$$\frac{\bar{u}}{R} = \frac{1}{2}(1-\eta^2) - \frac{k}{n+1}(1-\eta^{n+1}) + \frac{k^2}{2}(1-\eta^{2n})$$

$$- \frac{n}{2} k^3 (1-\eta^{3n-1}) + \frac{(4n^2-n)}{6} k^4 (1-\eta^{4n-2})$$

$$- \frac{(25n^3-15n^2+2n)}{24} k^5 (1-\eta^{5n-3}) \ldots \quad (6.4.9)$$

The total flux Q through the channel of width h is given by

$$Q = L . \int_0^h u . dy$$

which on simplification reduces to

$$Q' = \frac{Q}{ULh} = \frac{1}{3} - \frac{k}{n+2} + \frac{n}{2n+1} . k^2 - \frac{(3n-1)}{6} k^3$$

$$+ \frac{n(2n-1)}{3} k^4 - \frac{(5n-1)(5n-3)}{24} k^5 \cdots \quad (6.4.10)$$

6.5 SERIES SOLUTION WHEN μ IS SMALL

Assuming the series solution of the form

$$-\frac{\partial u}{\partial y} = \sum_{m=0}^{\infty} \delta_m \mu^m \qquad (6.5.1)$$

and substituting this value of $(-\partial u/\partial y)$ in (6.4.1) and collecting the coefficients of various powers of μ we get

$$\delta_0 = \left(\frac{P_y}{\mu_a}\right)^{\frac{1}{n}}$$

$$\delta_1 = -\frac{1}{n\mu_a}\left(\frac{P_y}{\mu_a}\right)^{\frac{2-n}{n}}$$

$$\delta_2 = \frac{(3-n)}{2n^2 \cdot \mu_a^2} \cdot \left(\frac{P_y}{\mu_a}\right)^{\frac{3-2n}{n}} \qquad (6.5.2)$$

$$\delta_3 = -\frac{(n^2-6n+8)}{3n^3\,\mu_a^3}\left(\frac{P_y}{\mu_a}\right)^{\frac{4-3n}{n}}$$

$$\delta_4 = -\frac{(6n^3-55n^2+150n-125)}{24\,n^4\,\mu_a^4}\left(\frac{P_y}{\mu_a}\right)^{\frac{5-4n}{n}}$$

and the series for $(\partial u/\partial y)$ is given by

$$\frac{\partial u}{\partial y} = -\left(\frac{P_y}{\mu_a}\right)^{\frac{1}{n}} + \frac{\mu}{n\mu_a}\left(\frac{P_y}{\mu_a}\right)^{\frac{2-n}{n}} - \frac{(3-n)}{2n^2}\cdot\frac{\mu^2}{\mu_a^2}\cdot\left(\frac{P_y}{\mu_a}\right)^{\frac{3-2n}{n}}$$

$$+ \frac{(n^2-6n+8)}{3n^3\,\mu_a^3}\cdot\mu^3\left(\frac{P_y}{\mu_a}\right)^{\frac{4-3n}{n}}$$

$$+ \frac{(6n^3-55n^2+150n-125)}{24n^4}\,\frac{\mu^4}{\mu_a^4}\left(\frac{P_y}{\mu_a}\right)^{\frac{5-4n}{n}} \cdots\cdots \qquad (6.5.3)$$

Integrating the above equation under the boundary condition that $u = 0$ at $y = h$ we get the following velocity profile

$$u = \frac{n}{n+1}\left(\frac{P}{\mu_a}\right)^{\frac{1}{n}}\left(h^{n+1} - y^{n+1}\right) - \frac{\mu}{2}\cdot\frac{P^{\frac{2-n}{n}}}{\mu_a^{2/n}}\left(h^{\frac{2}{n}} - y^{\frac{2}{n}}\right)$$

$$+ \frac{\mu^2}{2n}\,\frac{P^{\frac{3-2n}{n}}}{\mu_a^{3/n}}\cdot\left(h^{\frac{3-n}{n}} - y^{\frac{3-n}{n}}\right) \qquad (6.5.4)$$

$$+ \frac{(4-n)}{6n^2}\cdot\mu^3\cdot\frac{P^{\frac{4-3n}{n}}}{\mu_a^{4/n}}\left(h^{\frac{4-2n}{n}} - y^{\frac{4-2n}{n}}\right)$$

$$+ \frac{(2n^2 - 15n + 25)\mu^4}{24n^3}\cdot\frac{P^{\frac{5-4n}{n}}}{\mu_a^{5/n}}\left(h^{\frac{5-3n}{n}} - y^{\frac{5-3n}{n}}\right)\text{------}$$

Transforming the above equation into non-dimensional form as before with the help of substitutions (6.4.7) we get

$$\frac{\bar{u}}{K_o} = \frac{n}{n+1}\left(1 - \eta^{\frac{n+1}{n}}\right) - \frac{1}{2}\left(\frac{K_o}{R\bar{P}}\right)\left(1 - \eta^{2/n}\right)$$

$$+ \frac{1}{2n}\left(\frac{K_o}{R\bar{P}}\right)^2\left(1 - \eta^{\frac{3-n}{n}}\right) - \frac{(4-n)}{6n^2}\left(\frac{K_o}{R\bar{P}}\right)^3\left(1 - \eta^{\frac{4-2n}{n}}\right)$$

$$+ \frac{(2n^2 - 15n + 25)}{24n^3}\left(\frac{K_o}{R\bar{P}}\right)^4\left(1 - \eta^{\frac{5-3n}{n}}\right)\text{-----} \qquad (6.5.5)$$

where

$$K_o = \left(\frac{R\bar{P}}{R_1}\right)^{\frac{1}{n}}$$

The total flux Q_0 when μ is small is given by

$$Q' = \frac{Q_0}{ULh} = K_0 \left[\frac{n}{2n+1} - \left(\frac{K_0}{R\bar{P}}\right) \frac{1}{2+n} + \frac{3-n}{6n} \left(\frac{K_0}{R\bar{P}}\right)^2 \right.$$

$$\left. - \frac{(2-n)}{3n^2} \left(\frac{K_0}{R\bar{P}}\right)^3 + \frac{(5-3n)(5-n)}{24n^3} \left(\frac{K_0}{R\bar{P}}\right)^4 \cdots \right] \quad (6.5.6)$$

where h is the width of the channel.

6.6 FLOW THROUGH A CIRCULAR PIPE

Let us consider the same fluid (1.2.2) flowing through a circular pipe under constant pressure gradient along the axis of the pipe. The components of velocity are given by

$$u = 0 \quad , \quad v = 0 \qquad \text{and} \qquad w = w(r) \quad (6.6.1)$$

The components of the strain-rate are given by

$$e_{rr} = e_{\theta\theta} = e_{r\theta} = e_{\theta z} = e_{zz} = 0 \quad (6.6.2)$$

and
$$e_{rz} = \frac{1}{2} \frac{\partial w}{\partial r}$$

The components of the stress are than given by

$$t_{rr} = t_{\theta\theta} = t_{r\theta} = t_{\theta z} = t_{zz} = 0 \qquad (6.6.3)$$

and

$$t_{rz} = \frac{1}{2}\mu\left(\frac{\partial w}{\partial r}\right) + \frac{1}{2^n}\mu_a\left(\frac{\partial w}{\partial r}\right)^n$$

The equations of motion reduce to

$$\frac{\partial p}{\partial r} = 0$$

$$r\frac{\partial p}{\partial z} = \frac{\partial}{\partial r}(r.t_{rz}) \qquad (6.6.4)$$

The latter equation on integration reduces to :

$$-\left[\frac{dp}{dz}\cdot\frac{r^2}{2} + B\right] = r\left[\frac{\mu}{2}\cdot\left(\frac{\partial w}{\partial r}\right) + \frac{\mu_a}{2^n}\cdot\left(\frac{\partial w}{\partial r}\right)^n\right] \qquad (6.6.5)$$

Since at $r = 0$, $\frac{\partial w}{\partial r} = 0$ we get $B = 0$

and

$$-\frac{dp}{dz}\cdot\frac{r}{2} = \frac{\mu}{2}\left(-\frac{\partial w}{\partial r}\right) + \frac{\mu_a}{2^n}\left(-\frac{\partial w}{\partial r}\right)^n \qquad (6.6.6)$$

Since $(\partial w/\partial r)$ is essentially negative as w decreases as r increases.

This equation is essentially the same as derived in the case of flow between two plates considered in previous sections of the present chapter, except for the change of

fluid behaviour constants μ and μ_a and λ in place of y, and therefore we get the same solution in this case also.

6.7 DISCUSSIONS

The series for $\frac{\bar{u}}{R}$ given by the equation (6.4.9) is convergent for small R_1 and $n < 1$. For $n = 1$ we get $R_1 = k = \cdot 1$ (say), and this equation reduces to

$$\frac{\bar{u}}{R} = \frac{5}{11}\left(1-\eta^2\right) \tag{6.7.1}$$

which is the parabolic velocity profile for Newtonian fluid. For other values of n we can regard all the terms of (6.4.9) as the successive approximating to the velocity profile further approximations being very small. As illustrative examples we take $n = \frac{1}{2}$ and $k = \cdot 1$, the equation (6.4.9) can be written as :

$$\frac{\bar{u}}{R} = \frac{1}{2}\left(1-\eta^2\right) - \frac{1}{15}\left(1-\eta^{3/2}\right) + \frac{1}{200}\left(1-\eta\right) - \frac{1}{4000}\left(1-\eta^{1/2}\right) \cdots \tag{6.7.2}$$

For $n = \frac{1}{5}$ and $k = \cdot 1$, this reduces to

$$\frac{\bar{u}}{R} = \frac{1}{2}\left(1-\eta^2\right) - \frac{1}{12}\left(1-\eta^{6/5}\right) + \frac{1}{200}\left(1-\eta^{2/5}\right) - \frac{1}{10000}\left(1-\eta^{-2/5}\right) \cdots \tag{6.7.3}$$

The following table illustrates the successive approximations to the velocity profiles given by (6.7.2) and (6.7.3). From these we may conclude that further approximations will remain practically unaltered even if we take more and more terms of the velocity profile $\frac{\bar{u}}{R}$.

Table 6.1 : The successive approximations to the velocity profile (6.4.9) for $\kappa = .1$.

	η	First approximation	Second approximation	Third approximation	Fourth approximation
$n = 1/2$	0	.5000	.4333	.4383	.4382
	.2	.4800	.4198	.4233	.4232
	.4	.4200	.3702	.3732	.3731
	.6	.3200	.2863	.2883	.2883
	.8	.1800	.1610	.1620	.1620
	1.0	.0000	.0000	.0000	.0000
$n = 1/5$	0	.5000	.4167	.4216	.4215
	.2	.4800	.4038	.4111	.4111
	.4	.4200	.3488	.3530	.3530
	.6	.3200	.2818	.2827	.2827
	.8	.1800	.1604	.1598	.1598
	1.0	.0000	.0000	.0000	.0000

P A R T   T W O

CHAPTER VII

STREAMLINE FLOW THROUGH CURVED ANNULUS

7.1 INTRODUCTION

The problem of the flow of fluids in a curved channel is of great importance, both from the biologists and engineers point of view. We have already given in section 1.7 a brief account of all the investigations on flow in curved channels. In the present chapter, we shall extend the results of Dean (1927) for a curved pipe to the case of a curved annulus. In section 7.2 we shall describe the geometrical problem. In section 7.3 we shall give the mathematical formulation of the same. In the next three sections 7.4 - 7.6 we shall obtain expressions for ψ', ω' and p' which determine the secondary flow. In section 7.7 and 7.8 we shall give the expression for the stress-components. In the next three sections

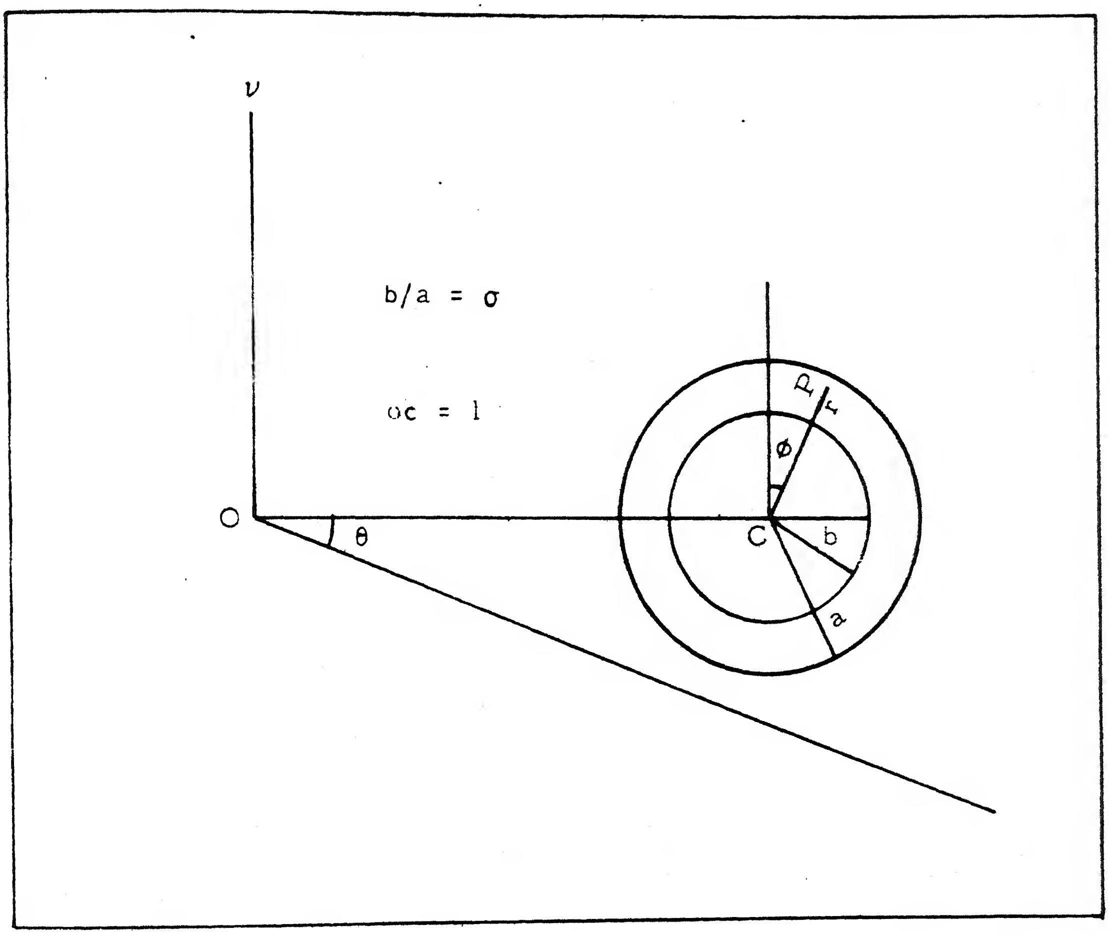

Figure 7.1 The coordinate system (r, ϕ, θ) is chosen to describe motion in a curved annulus of circular cross-section.

(7.9 - 7.11) we give a numerical example and for this example we obtain the projection of stream lines on the normal section and the flow line on the central plane. In the last section we shall be giving the effect of radii ratio σ on the velocity field.

7.2 GEOMETRY OF THE PROBLEM

Our annulus is the region between two co-axial pipes of circular cross-sections. The radius of the outer pipe is a and that of the inner pipe is b. The line of centres of the annular cross-sections is coiled in a circle with centre O and radius l. This line is called the central line. The coordinate system used is that adopted by Dean (1927) in discussing the curved pipe flow problem and is represented in figure (7.1). $O\nu$ is an axis through O and perpendicular to the plane (called the central plane) of the central line of the annulus, C is the centre of the annular cross-section by a plane passing through $O\nu$ making an angle θ with a fixed plane $(\theta = 0)$ through $O\nu$. OC is perpendicular to $O\nu$ and is of length l. Any point P, of the annular cross-section θ = constant is referred to by the orthogonal curvilinear coordinates r, ϕ and θ, where r is the distance CP and ϕ is the angle made by CP with the line through C parallel to $O\nu$. The line element is then given by

$$(ds)^2 = (dr)^2 + (r\,d\phi)^2 + (l + r\sin\phi)^2 (d\theta)^2 \qquad (7.2.1)$$

7.3 MATHEMATICAL FORMULATION

In the curved pipes, there exists a secondary flow (Barua (1964) and others). The same will also exist in the curved annulus, because the fluid particles near the central portion of the flow region which have a higher velocity, are acted upon by a larger centrifugal force than the slower particles near the walls. This leads to the emergence of a secondary flow which is directed out-wards (i.e. away from the centre of curvature) near the maximum velocity region and near the inner pipe wall and is directed inwards (i.e. towards the centre of curvature) near the outer cylinder wall. So in the light of the above physical facts, the resultant flow should be three dimensional. In the case of regular flow (i.e. laminar flow), the flow field will be symmetrical about the central plane (i.e. the plane whose normal is $O\nu$ and which passes through Oc .

We proceed to discuss the steady and fully developed motion so that the velocity components in the coordinate directions r , θ , ϕ may be written as $U(r,\phi)$, $V(r,\phi)$ and $W(r,\phi)$, all independent of θ . The rate-of-strain tensor has then the physical components

$$e_{rr} = \frac{\partial U}{\partial r}$$

$$e_{\phi\phi} = \frac{1}{r}\cdot\frac{\partial V}{\partial \phi} + \frac{U}{r}$$

$$e_{\theta\theta} = \frac{U \sin\phi + V\cos\phi}{l + r\sin\phi}$$

$$e_{\phi\theta} = \frac{1}{2}\left(\frac{1}{r}\cdot\frac{\partial W}{\partial \phi} - \frac{W\cos\phi}{l + r\sin\phi} \right)$$

$$e_{\theta r} = \frac{1}{2}\left(\frac{\partial W}{\partial r} - \frac{W\sin\phi}{l + r\sin\phi} \right) \qquad (7.3.1)$$

$$e_{r\phi} = \frac{1}{2}\left[r\frac{\partial}{\partial r}\left(\frac{V}{r}\right) + \frac{1}{r}\cdot\frac{\partial V}{\partial \phi} \right]$$

From the equation of state for laminar, incompressible, Newtonian fluid :

$$t_{ij} = -P\delta_{ij} + 2\mu\, e_{ij} \qquad (7.3.2)$$

the stress tensor t_{ij} **has the physical components :**

$$t_{rr} = -P + 2\mu\frac{\partial U}{\partial r}$$

$$t_{\phi\phi} = -P + 2\mu\left(\frac{1}{r}\frac{\partial V}{\partial \phi} + \frac{U}{r} \right)$$

$$t_{\theta\theta} = -P + 2\mu\left(\frac{U\sin\phi + V\cos\phi}{l + r\sin\phi} \right) \qquad (7.3.3)$$

$$t_{\phi\theta} = \mu\left(\frac{1}{r}\frac{\partial W}{\partial \phi} - \frac{W\cos\phi}{l + r\sin\phi} \right)$$

$$t_{\phi r} = \mu\left(\frac{\partial W}{\partial r} - \frac{W\sin\phi}{l + r\sin\phi} \right)$$

$$t_{r\phi} = \mu\left[r\frac{\partial}{\partial r}\left(\frac{V}{r}\right) + \frac{1}{r}\frac{\partial V}{\partial \phi} \right]$$

Here t_{ij} and e_{ij} are the stress and rate-of-strain tensor, p is an undetermined isotropic pressure, δ_{ij} is the Kronecker tensor and μ is the coefficient of viscosity. We assume that the fluid properties are physically constant quantities and the fluid motion is a forced motion.

In absence of the body forces, the equations of linear momentum are :

$$\rho\left(U\frac{\partial U}{\partial r}+\frac{V}{r}\frac{\partial U}{\partial \phi}-\frac{V^2}{r}-\frac{W^2\sin\phi}{1+r\sin\phi}\right)=\frac{\partial t_{rr}}{\partial r}+\frac{t_{rr}-t_{\phi\phi}}{r}$$
$$+\frac{t_{r\phi}\cos\phi}{1+r\sin\phi}+\frac{1}{r}\frac{\partial t_{r\phi}}{\partial \phi}+\frac{(t_{rr}-t_{\theta\theta})\sin\phi}{1+r\sin\phi}$$
$$+\frac{1}{1+r\sin\phi}\cdot\frac{\partial t_{\theta r}}{\partial \theta} \qquad (7.3.4)$$

$$\rho\left(U\frac{\partial V}{\partial r}+\frac{V}{r}\frac{\partial V}{\partial \phi}+\frac{UV}{r}-\frac{W^2\cos\phi}{1+r\sin\phi}\right)=\frac{\partial t_{r\phi}}{\partial r}+\left(\frac{2}{r}+\frac{\sin\phi}{1+r\sin\phi}\right)t_{r\phi}$$
$$+\frac{1}{r}\frac{\partial t_{\phi\phi}}{\partial \phi}+\frac{(t_{\phi\phi}-t_{\theta\theta})\cos\phi}{1+r\sin\phi}$$
$$+\frac{1}{1+r\sin\phi}\frac{\partial t_{\theta r}}{\partial \theta} \qquad (7.3.5)$$

$$\rho\left(U\frac{\partial W}{\partial r}+\frac{V}{r}\frac{\partial W}{\partial \phi}+\frac{UW\sin\phi+VW\cos\phi}{1+r\sin\phi}\right)=\frac{\partial t_{\theta r}}{\partial r}+\left(\frac{1}{r}+\frac{2\sin\phi}{1+r\sin\phi}\right)t_{\theta r}$$
$$+\frac{1}{r}\frac{\partial t_{\phi\theta}}{\partial \phi}+\frac{2t_{\phi\theta}\cos\phi}{1+r\sin\phi}+\frac{1}{1+r\sin\phi}\frac{\partial t_{\theta\theta}}{\partial \theta} \qquad (7.3.6)$$

The equation of continuity is

$$\frac{\partial U}{\partial r} + \frac{U}{r} + \frac{1}{r}\frac{\partial V}{\partial \phi} + \frac{U \sin\phi + V\cos\phi}{l + r\sin\phi} = 0 \tag{7.3.7}$$

If the annulus were straight $1/l$ would vanish, and (7.3.3) to (7.3.7) would be satisfied by the expressions :

$$W = \frac{Aa^2}{4\mu}\left[1 - \left(\frac{r}{a}\right)^2 + \frac{1-\sigma^2}{\ln\left(\frac{1}{\sigma}\right)}\ln\left(\frac{r}{a}\right)\right]$$

and $$P = -Az + B \tag{7.3.8}$$

together with $U=0$, $V=0$, and $\sigma = \frac{b}{a}$.

We assume, for the purpose of mathematical analysis, that the annulus is slightly curved, that is $\frac{a}{l}$ and $\frac{b}{l}$ are small quantities of the first order. Following Dean (1927) we write

$$U = u \;,\quad V = v \;,\quad W = w_1 + w \quad \text{and} \quad P = p_1 + p \tag{7.3.9}$$

where u, v, w and p are referred to secondary flow and are taken to be small of the order of $\frac{a}{l}$ and also u, v and w are independent of θ . w_1 and p_1 are the velocity and the pressure terms for the corresponding problem for straight annulus. We know that :

$$\omega_1 = \frac{Aa^2}{4\mu}\left[1-\left(\frac{r}{a}\right)^2-\frac{1-\sigma^2}{\ln\sigma}\ln\left(\frac{r}{a}\right)\right]$$

$$p_1 = -Az + B \qquad (7.3.10)$$

Neglecting terms of order $\left(\frac{a}{\ell}\right)^2$.. the equation of continuity becomes

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \phi} = 0 \qquad (7.3.11)$$

This suggests the stream function ψ of the form :

$$u = -\frac{1}{r}\frac{\partial \psi}{\partial \phi}, \quad v = \frac{\partial \psi}{\partial r} \qquad (7.3.12)$$

Substituting (7.3.9), (7.3.10) and (7.3.3) in the equations of motion (7.3.4) to (7.3.6) and neglecting the small terms and simplifying to a convenient form we find that (7.3.4) to (7.3.6) reduce to :

$$-\frac{\rho A^2 a^4}{16\mu^2 \ell}\left(1-\xi^2-\lambda\ln\xi\right)^2\sin\phi = -\frac{\partial p}{\partial r}$$

$$-\frac{\mu}{r}\frac{\partial}{\partial\phi}\left(\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\phi^2}\right) \qquad (7.3.13)$$

$$-\frac{\rho A^2 a^4}{16\mu^2 \ell}\left(1-\xi^2-\lambda\ln\xi\right)^2\cos\phi = -\frac{1}{r}\frac{\partial p}{\partial\phi}$$

$$+\mu\frac{\partial}{\partial r}\left(\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\phi^2}\right) \qquad (7.3.14)$$

$$\rho A\left(2\varsigma + \frac{\lambda}{\varsigma}\right)\frac{\partial\psi}{\partial\phi} + \frac{Aa}{4l}\left(6\varsigma + \frac{\lambda}{\varsigma}\right)\sin\phi = -\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial\phi^2}\right) \tag{7.3.15}$$

where

$$\varsigma = r/a \quad ; \quad \lambda = \frac{1-\sigma^2}{\ln\sigma}$$

$$\sigma = b/a \quad ; \quad z = l\theta \tag{7.3.16}$$

Since ψ and w are independent of θ, (7.3.15) suggests that we can take

$$\frac{\partial p}{\partial z} = \frac{1}{l}\cdot\frac{\partial p}{\partial\theta} = 0 \tag{7.3.17}$$

The physical picture of the problem and equations (7.3.13), (7.3.14) and (7.3.15) suggest that ψ, w and p should be of the forms

$$\psi = \frac{Aa^4}{4\mu l}\cdot\psi'(\varsigma)\cos\phi$$

$$w = \frac{Aa^3}{4\mu l}\cdot w'(\varsigma)\sin\phi \tag{7.3.18}$$

and

$$p = \frac{Aa^2}{4l}\cdot p'(\varsigma)\sin\phi$$

where ψ', ω' and p' are functions of $\xi = \frac{r}{a}$ only.
Substituting for ψ, ω and p in terms of the dimensionless functions ψ', ω' and p' in (7.3.13) to (7.3.15), the following equations are obtained

$$-R\left(1-\xi^2-\lambda\ln\xi\right)^2 = \frac{dp'}{d\xi} + \frac{1}{\xi}\Delta\psi' \tag{7.3.19}$$

$$-R\left(1-\xi^2-\lambda\ln\xi\right)^2 = -\frac{p'}{\xi} + \frac{d}{d\xi}\left(\Delta\psi'\right) \tag{7.3.20}$$

$$-R\,\frac{1}{\xi}\left(2\xi+\frac{\lambda}{\xi}\right)\psi' + \left(6\xi+\frac{\lambda}{\xi}\right) = \Delta\omega' \tag{7.3.21}$$

where R is the Reynolds number given by

$$R = \frac{\rho A a^3}{4\mu^3} \tag{7.3.22}$$

and Δ is the operator defined by

$$\Delta \equiv \frac{d^2}{d\xi^2} + \frac{1}{\xi}\frac{d}{d\xi} - \frac{1}{\xi^2} \tag{7.3.23}$$

Eliminating p' between (7.3.19) and (7.3.20) we get

$$2R\left(2\xi+\frac{\lambda}{\xi}\right)\left(1-\xi^2-\lambda\ln\xi\right) = \Delta\left(\Delta\psi'\right) \tag{7.3.24}$$

7.4 SOLUTION FOR $\psi'(\xi)$

Equation (7.3.24) can be written as :

$$\frac{d}{d\xi}\left[\frac{1}{\xi}\frac{d}{d\xi}\{\xi(\Delta\psi')\}\right] = 2R(2-\lambda)\xi - 4R\xi^3$$

$$+ 2R\lambda\frac{1}{\xi^2} - 4R\lambda\xi\ln\xi - 2R\lambda^2\frac{1}{\xi}\ln\xi \qquad (7.4.1)$$

Integrating (7.4.1) we obtain

$$\Delta\psi' = \left(\tfrac{1}{2}E - \tfrac{1}{4}R\lambda^2 - \tfrac{1}{2}R\lambda\right)\xi + \left(\tfrac{1}{2}R + \tfrac{1}{8}R\lambda\right)\xi^3$$

$$- \tfrac{1}{6}R\xi^5 + \left(R\lambda + \tfrac{1}{2}R\lambda^2\right)\xi\ln\xi - \tfrac{1}{2}R\lambda\xi^3\ln\xi$$

$$- \tfrac{1}{2}R\lambda^2\xi(\ln\xi)^2 + \frac{F}{\xi} \qquad (7.4.2)$$

$$\psi' = \frac{H}{\xi} + \left(\tfrac{1}{2}G - \tfrac{1}{4}F\right)\xi + \left(\tfrac{1}{16}E - \tfrac{5}{32}R\lambda - \tfrac{17}{128}R\lambda^2\right)\xi^3$$

$$+ \left(\tfrac{1}{48}R + \tfrac{1}{72}R\lambda\right)\xi^5 - \tfrac{1}{288}R\xi^7 - \tfrac{1}{48}R\lambda\xi^5\ln\xi$$

$$+ \left(\tfrac{1}{8}R\lambda + \tfrac{5}{32}R\lambda^2\right)\xi^3\ln\xi + \tfrac{1}{2}F\xi\ln\xi$$

$$- \frac{1}{16}R\lambda^2\xi^3(\ln\xi)^2 \qquad (7.4.3)$$

Differentiating each term of (7.4.3) we obtain

$$\frac{d\psi'}{d\xi} = -\frac{H}{\xi^2} + \left(\frac{1}{2}G + \frac{1}{4}F\right) + \frac{3}{16}E\xi^2$$

$$-\left(\frac{11}{32}R\lambda + \frac{31}{128}R\lambda^2\right)\xi^2 + \left(\frac{5}{48}R + \frac{7}{144}R\lambda\right)\xi^4$$

$$-\frac{7}{288}R\xi^6 - \frac{5}{48}R\lambda\xi^4 \ln\xi + \left(\frac{3}{8}R\lambda + \frac{11}{32}R\lambda^2\right)\xi^2 \ln\xi$$

$$+\frac{1}{2}F\ln\xi - \frac{3}{16}R\lambda^2\xi^2(\ln\xi)^2$$

(7.4.4)

Here E, F, G and H are constants of integration. In order to determine these four constants we use the following boundary conditions :

$$\psi'(\xi) = \frac{d\psi'}{d\xi} = 0 \quad \text{at} \quad \xi = 1 \tag{7.4.5}$$

$$\psi'(\xi) = \frac{d\psi'}{d\xi} = 0 \quad \text{at} \quad \xi = \sigma \tag{7.4.6}$$

The boundary condition $\psi'(1) = 0$ gives

$$2E - 8F + 16G + 32H = \frac{17}{4}R\lambda^2 + \frac{41}{9}R\lambda - \frac{5}{9}R = L_1 \text{ (say)} \tag{7.4.7}$$

The boundary condition $\left.\frac{d\psi'}{d\xi}\right|_{\xi=1} = 0$ gives

$$6E + 8F + 16G + 32H = \frac{31}{4} R\lambda^2 + \frac{85}{9} R\lambda - \frac{23}{9} R = L_2 \quad \textbf{(say)(7.4.8)}$$

The boundary condition $\psi'(\sigma) = 0$ gives

$$2\sigma^4 E + (16\sigma^2 \ln\sigma - 8\sigma^2)F + 16\sigma^2 G + 32H$$

$$= \left\{ \frac{17}{4}\sigma^4 - 5\sigma^4 \ln\sigma + 2\sigma^4 (\ln\sigma)^2 \right\} R\lambda^2$$

$$+ \left\{ 5\sigma^4 - \frac{4}{9}\sigma^6 + \frac{2}{3}\sigma^6 \ln\sigma - 4\sigma^4 \ln\sigma \right\} R\lambda$$

$$+ \left(\frac{1}{9}\sigma^8 - \frac{2}{3}\sigma^6 \right) R \qquad = L_3 \qquad \textbf{(say)} \qquad \textbf{(7.4.9)}$$

And the boundary condition $\left.\frac{d\psi'}{d\rho}\right|_{\rho=\sigma} = 0$ gives

$$6\sigma^4 E + (8\sigma^2 + 16\sigma^2 \ln\sigma)F + 16\sigma^2 G - 32H$$

$$= \left\{ \frac{31}{4}\sigma^4 - 11\sigma^4 \ln\sigma + 6\sigma^4 (\ln\sigma)^2 \right\} R\lambda^2$$

$$+ \left\{ 11\sigma^4 - \frac{14}{9}\sigma^6 + \frac{10}{3}\sigma^6 \ln\sigma - 12\sigma^4 \ln\sigma \right\} R\lambda$$

$$+ \left(\frac{7}{9}\sigma^8 - \frac{10}{3}\sigma^6 \right) R \quad = L_4 \quad \textbf{(say)} \qquad \textbf{(7.4.10)}$$

Solving (7.4.7) to (7.4.10) we find that

$$E = \frac{1}{\left[8\sigma^2(1-\sigma^2)\{(1+\sigma^2)\ln\sigma + (1-\sigma^2)\}\right]} \times$$

$$\left[\sigma^2\{(1-\sigma^2) - 2\ln\sigma\}L_1 + \sigma^2\{2\ln\sigma + (1-\sigma^2)\}L_2\right.$$

$$\left. + \{2\sigma^2\ln\sigma - (1-\sigma^2)\}L_3 - \{2\sigma^2\ln\sigma + (1-\sigma^2)\}L_4\right] \quad (7.4.11)$$

$$F = \left[(1-\sigma^2)L_4 + (1+3\sigma^2)L_3 + \sigma^2(1-\sigma^2)L_2\right.$$

$$\left. - \sigma^2(3+\sigma^2)L_1\right] \Big/$$

$$\left[32\sigma^2\{(1+\sigma^2)\ln\sigma + (1-\sigma^2)\}\right] \quad (7.4.12)$$

$$G = \frac{1}{32}(L_1 + L_2) - \frac{1}{4}E \quad (7.4.13)$$

$$H = \frac{1}{16}E + \frac{1}{4}F + \frac{1}{64}(L_1 - L_2) \quad (7.4.14)$$

Now substituting the values of E, F, G **and** H **from (7.4.11) to (7.4.14) in (7.4.3) and (7.4.4) we can completely determine** ψ' **and** $\frac{d\psi'}{ds}$ **and hence** u **and** v **for arbitrary value of radii ratio** σ **.**

7.5 SOLUTION FOR $\omega'(\xi)$

In order to calculate ω' we rewrite (7.3.21) in the following form :

$$\frac{d}{d\xi}\left[\frac{1}{\xi}\frac{d}{d\xi}(\xi\omega')\right] = 6\xi + \frac{\lambda}{\xi} - 2R\chi' - R\lambda\frac{1}{\xi^2}\psi' \qquad (7.5.1)$$

Substituting the value of ψ' from (7.4.3) in (7.5.1) and then integrating we get

$$\begin{aligned}
\omega'(\xi) = {} & \frac{J}{\xi} + \left(\tfrac{1}{2}I - \tfrac{1}{8}R\lambda F - \tfrac{1}{4}\lambda + \tfrac{1}{2}RH + \tfrac{1}{8}R\lambda G\right)\xi + \\
& \left(\tfrac{3}{4} - \tfrac{1}{8}RG + \tfrac{5}{32}RF - \tfrac{1}{128}R\lambda E + \tfrac{1}{32}R^2\lambda^2 + \tfrac{39}{1024}R^2\lambda^3\right)\xi^3 \\
& + \left(\tfrac{117}{6912}R^2\lambda^2 + \tfrac{19}{3072}R^2\lambda - \tfrac{1}{192}RE\right)\xi^5 + \tfrac{1}{11520}R^2\xi^9 \\
& - \left(\tfrac{1}{1152}R^2 + \tfrac{7}{9216}R^2\lambda\right)\xi^7 + \tfrac{1}{1152}R^2\xi^7\lambda\ln\xi \\
& - \left(\tfrac{1}{96}R^2\lambda + \tfrac{19}{3072}R^2\lambda^2\right)\xi^5\ln\xi \\
& - \left(\tfrac{1}{8}RF + \tfrac{1}{64}R^2\lambda^2 + \tfrac{1}{32}R^2\lambda^3\right)\xi^3\ln\xi \\
& + \left(\tfrac{1}{2}\lambda - RH - \tfrac{1}{4}R\lambda G + \tfrac{1}{4}R\lambda F\right)\xi\ln\xi \\
& + \tfrac{1}{2}R\lambda H\,\tfrac{1}{\xi}\ln\xi + \tfrac{1}{192}\cdot R^2\lambda^2\cdot\xi^5(\ln\xi)^2 \\
& + \tfrac{1}{128}R^2\lambda^3\xi^3(\ln\xi)^2 - \tfrac{1}{8}R\lambda F\xi(\ln\xi)^2
\end{aligned} \qquad (7.5.2)$$

where I and J are constants of integration.

To evaluate these constants we use the following boundary conditions :

$$\omega'(1) = 0 \quad , \quad \omega'(\sigma) = 0 \qquad (7.5.3)$$

The boundary condition $\omega'(1) = 0$ gives

$$-J - \frac{1}{2} I = \left(-\frac{1}{8} R\lambda F - \frac{1}{4}\lambda + \frac{1}{2} RH + \frac{1}{8} R\lambda G\right)$$

$$+\left(\frac{3}{4} - \frac{1}{8} RG + \frac{5}{32} RF - \frac{1}{128} R\lambda E + \frac{1}{32} R^2\lambda^2 + \frac{39}{1024} R^2\lambda^3\right)$$

$$+ \left(\frac{117}{6912} R^2\lambda^2 + \frac{19}{3072} R^2\lambda - \frac{1}{192} RE\right)$$

$$- \left(\frac{1}{1152} R^2 + \frac{7}{9216} R^2\lambda\right) + \frac{1}{11520} R^2$$

$$= M_1 \qquad \text{(say)} \qquad (7.5.4)$$

and the boundary condition $\omega'(\sigma) = 0$ gives

$$-J - \frac{1}{2} I \sigma^2 = \left(-\frac{1}{8} R\lambda F - \frac{1}{4}\lambda + \frac{1}{2} RH + \frac{1}{8} R\lambda G\right)\sigma^2$$

$$+ \left(\frac{3}{4} - \frac{1}{8} RG + \frac{5}{32} RF - \frac{1}{128} R\lambda E + \frac{1}{32} R^2\lambda^2 + \frac{39}{1024} R^2\lambda^3\right)$$

$$+ \left(\frac{117}{6912} R^2\lambda^2 + \frac{19}{3072} R^2\lambda - \frac{1}{192} RE \right) \sigma^6$$

$$- \left(\frac{1}{1152} R^2 + \frac{7}{9216} R^2\lambda \right) \sigma^8 + \frac{1}{11520} R^2\sigma^{10}$$

$$+ \frac{1}{1152} R^2\lambda\sigma^8 \ln\sigma - \left(\frac{1}{96} R^2\lambda + \frac{19}{1152} R^2\lambda^2 \right) \sigma^6 \ln\sigma$$

$$- \left(\frac{1}{8} RF + \frac{1}{64} R^2\lambda^2 + \frac{1}{32} R^2\lambda^3 \right) \sigma^4 \ln\sigma$$

$$+ \left(\frac{1}{2}\lambda - RH - \frac{1}{4} R\lambda G + \frac{1}{4} R\lambda F \right) \sigma^2 \ln\sigma$$

$$+ \frac{1}{2} R\lambda H \ln\sigma + \frac{1}{192} R^2\lambda^2\sigma^6 (\ln\sigma)^2$$

$$+ \frac{1}{128} R^2\lambda^3 \sigma^4 (\ln\sigma)^2$$

$$- \frac{1}{8} R\lambda F\sigma^2 (\ln\sigma)^2$$

$$= M_2 \qquad \text{(say)} \qquad (7.5.5)$$

Solving (7.5.4) and (7.5.5) we obtain

$$I = \frac{2}{(1-\sigma^2)} (M_2 - M_1) \qquad (7.5.6)$$

$$J = -\frac{1}{(1-\sigma^2)} (M_1 \sigma^2 - M_2) \qquad (7.5.7)$$

Thus calculating ω' we have found the axial component of the secondary flow.

7.6 SOLUTION FOR p'

(1.3.20) can be rewritten as :

$$\frac{p'}{\rho} = \frac{d}{d\rho}(\Delta\psi') + R(1-\rho^2-\lambda \ln\rho)^2 \qquad (7.6.1)$$

Differentiating both sides of (7.4.2) with respect to ρ and then substituting the expression for $\frac{d}{d\rho}(\Delta\psi')$ in (7.6.1) we find

$$p' = -\frac{E}{\rho} + \left(\frac{1}{2}E + \frac{1}{2}R\lambda + \frac{1}{4}R\lambda^2\right)\rho + \left(\frac{3}{2}R - \frac{1}{8}R\lambda\right)\rho^3$$

$$-\frac{5}{6}R\rho^5 - \frac{3}{2}R\lambda\rho^3\ln\rho + \left(R\lambda - \frac{1}{2}R\lambda^2\right)\rho\ln\rho$$

$$-\frac{1}{2}R\lambda^2\rho(\ln\rho)^2 + R\rho(1-\rho^2-\lambda\ln\rho)^2 \qquad (7.6.2)$$

7.7 EXPRESSION FOR $\tau_{\theta r}$

The frictional force per unit area of the external boundary in the axial direction is given by

$$t_{\theta r}\Big|_{r=a} = \mu\left[\frac{dw_1}{ds}\cdot\frac{1}{a} + \frac{1}{a}\cdot\left(\frac{\partial w}{\partial s}\right) - \frac{w_1 \sin\phi}{\ell}\right]_{s=1}$$

$$= -\frac{Aa}{4}\left[2+\lambda - \frac{a}{\ell}\cdot\left(\frac{dw'}{ds}\right)_{s=1}\sin\phi\right] \qquad (7.7.1)$$

and the frictional force per unit area of the inner boundary is given by

$$t_{\theta r}\Big|_{r=b} = -\frac{Aa}{4}\left[2\sigma + \frac{\lambda}{\sigma} - \frac{a}{\ell}\cdot\left(\frac{dw'}{ds}\right)_{s=\sigma}\sin\phi\right] \qquad (7.7.2)$$

Here $\frac{dw'}{ds}$ is to be calculated from (7.5.2).

7.8 EXPRESSION FOR $t_{r\phi}$

The frictional force per unit area of the outer surface in the circumferential direction is given by

$$t_{r\phi}\Big|_{r=a} = \mu\left[r\frac{d}{dr}\left(\frac{v}{r}\right) + \frac{1}{r}\frac{\partial u}{d\phi}\right]_{r=a}$$

$$= \frac{Aa^2}{4\ell}\left[\frac{d^2\psi'}{ds^2} - \frac{1}{s}\frac{d\psi'}{ds} + \frac{\psi'}{s^2}\right]_{s=1}\cos\phi \qquad (7.8.1)$$

and the frictional force per unit area of the inner surface in the circumferential direction is given by

$$t_{r\phi}\Big|_{r=b} = \frac{Aa^2}{4\ell}\left[\frac{d^2\psi'}{ds^2} - \frac{1}{s}\cdot\frac{d\psi'}{ds} + \frac{\psi'}{s^2}\right]_{s=\sigma}\cos\phi \qquad (7.8.2)$$

Here ψ' and $\frac{d\psi'}{d\xi}$ are given by (7.4.3) and (7.4.4) respectively, also the value of $\frac{d^2\psi'}{d\xi^2}$ can be calculated by differentiating (7.4.4) once with respect to ξ .

Substituting $1/\ell = 0$ in (7.8.1) and (7.8.2) we obtain

$$t_{r\phi}\Big|_{r=a} = t_{r\phi}\Big|_{r=b} = 0$$

that is, the case of straight annulus.

7.9 NUMERICAL EXAMPLE

So far we have found the various flow parameters for arbitrary value of the ratio $\sigma = \frac{b}{a}$. Now we proceed to discuss a particular case for which the radii-ratio is $\frac{1}{2}$, that is,

$$\sigma = 1/2 \tag{7.9.1}$$

so that

$$\ln\sigma = \ln(1/2) = -0.6931$$

$$(\ln\sigma)^2 = .480388 \tag{7.9.2}$$

$$\lambda = -1.082095,\ \lambda^2 = 1.17093,\ \lambda^3 = -1.267057$$

$$\begin{aligned} L_1 &= -0.508647\,R, & E &= -0.989285\,R, \\ L_2 &= -3.700634\,R, & F &= 0.048879\,R, \\ L_3 &= 0.114647\,R, & G &= 0.115781\,R, \\ L_4 &= 0.04594\,R, & H &= .000265\,R. \end{aligned} \tag{7.9.3}$$

$$M_1 = 1.020524 - 0.028614\, R^2$$

$$M_2 = 0.208256 - 0.006700\, R^2$$

$$I = 0.058437\, R^2 - 2.166048 \qquad (7.9.4)$$

$$J = 0.062500 - 0.000605\, R^2$$

Using these numerical values we obtain

$$\begin{aligned}\frac{\psi'}{R} = {} & 0.000265 \frac{1}{\xi} + 0.045671\, \xi - 0.048267\, \xi^3 \\ & + 0.005804\, \xi^5 - 0.003472\, \xi^7 + 0.022544\, \xi^5 \ln\xi \\ & + 0.047696\, \xi^3 \ln\xi + 0.02444 \ln\xi \\ & - 0.073183\, \xi^3 (\ln\xi)^2 \end{aligned} \qquad (7.9.5)$$

$$\begin{aligned}\frac{1}{R}\frac{d\psi'}{d\xi} = {} & -0.000265 \frac{1}{\xi^2} + 0.070110 - 0.097105\, \xi^2 \\ & + 0.051565\, \xi^4 - 0.024306\, \xi^6 + 0.112718\, \xi^4 (\ln\xi) \\ & - 0.003278\, \xi^2 (\ln\xi) + 0.024434 \ln\xi \\ & - 0.219549\, \xi^2 (\ln\xi)^2 \end{aligned} \qquad (7.9.6)$$

$$\begin{aligned}\omega'(\xi) = {} & (0.0625 - 0.000605\, R^2) \frac{1}{\xi} \\ & + (0.020302\, R^2 - 0.812500)\, \xi + (0.75 - 0.026864\, R^2)\, \xi^3 \\ & + 0.007126\, \xi^5 R^2 - 0.000046\, R^2 \xi^7 + 0.000087\, R^2 \xi^9 \\ & - 0.000939\, R^2 \xi^7 (\ln\xi) - 0.00804\, R^2 \xi^5 (\ln\xi) \\ & + 0.151899\, R^2 \xi^3 (\ln\xi) + (0.017834\, R^2 - 0.541048)\, \xi \ln\xi \\ & - 0.000144\, R^2 \frac{1}{\xi} \ln\xi + 0.006099\, R^2 \xi^5 (\ln\xi)^2 \\ & - 0.009899\, R^2 \xi^3 (\ln\xi)^2 + 0.006612\, \xi (\ln\xi)^2 \end{aligned} \qquad (7.9.7)$$

$$\frac{p'}{R} = -0.048879\,\frac{1}{\rho} - 0.742958\,\rho + 1.635262\,\rho^3$$
$$-0.833333\,\rho^5 + 1.623143\,\rho^3(\ln\rho) - 0.49663\,\rho(\ln\rho)$$
$$-0.585465\,\rho(\ln\rho)^2 + \rho(1-\rho^2+1.082095\,\ln\rho)^2 \qquad (7.9.8)$$

7.10 PROJECTION OF THE STREAM LINES ON A NORMAL SECTION

Dean (1927) has drawn the curves of intersection of the surfaces ψ = constant with a normal section θ =constant for the flow of an ordinary viscous liquid through a curved pipe and Jones (1960) has done this for the case of non-Newtonian liquids. We now do this for the present case. The curves have the polar equation

$$\sin\phi = k\left(\frac{\psi'}{R}\right) \qquad (7.10.1)$$

where ψ'/R (which is independent of R) is given by (7.4.3), and k is an arbitrary constant. Since k takes arbitrary numerical values, the relation between ρ and ϕ expressed by (7.10.1) depends neither on $\frac{a}{l}$ nor on R. For the case of $\sigma=\frac{1}{2}$, (7.10.1) takes the form

$$\sec\phi = \sigma'\,[\,0.000265/\rho + 0.045671\,\rho - 0.048267\,\rho^3$$
$$+ 0.005804\,\rho^5 - 0.003472\,\rho^7 + 0.022544\,\rho^5(\ln\rho)$$
$$+ 0.047696\,\rho^3(\ln\rho) + 0.02444\,\rho(\ln\rho)$$
$$- 0.073183\,\rho^3(\ln\rho)^2\,] \qquad (7.10.2)$$

We shall use σ' which is given by

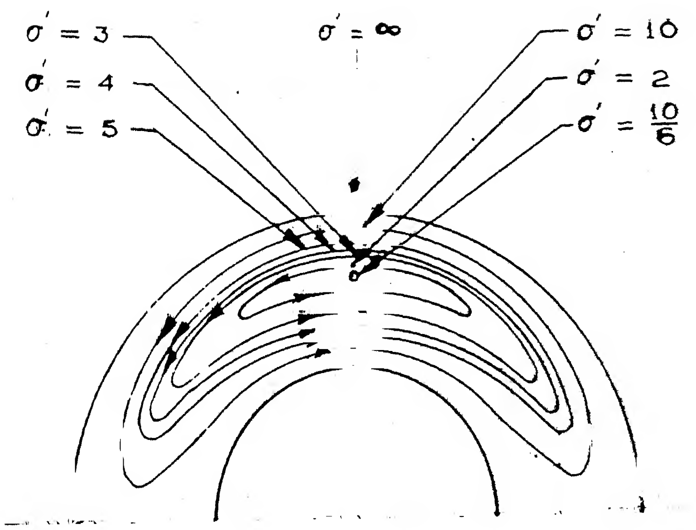

Figure 7.2 Curves of intersection of the surfaces
ψ = constant with a normal section
θ = constant.

$$\sigma = 100000 \; \sigma' \tag{7.10.3}$$

Calculating the values of ψ'/R (defined by (7.9.5)) for various values of ξ in the closed interval $0.5 \leqslant \xi \leqslant 1.0$ we found that the function ψ'/R attains its maximum value nearly at $\xi = 0.80$ (independently of Reynolds number). This means that $(1/R)(d\psi'/d\xi) = 0$ at $\xi = 0.80$. Thus at the points ($\xi = 0.080$, $\phi = n\pi$), the velocity component V vanishes. For every particular value of σ , there is therefore a limiting surface $\psi = \psi_m$ (the constant ψ_m depends on σ) which takes the degenerate form of a single circular streamline in a plane parallel to the central plane. The intersection of the stream line $\psi = \psi_m$ with a normal section θ = constant is a point. The stream line $\psi = \psi_m$ is defined by $\sigma' = 10/6$ when $\sigma = 1/2$ and the corresponding point of intersection with the section θ = constant is shown in fig. (7.2) $\sigma' = \infty$, for all values of σ , corresponds to the annulus-walls. For the case of $\sigma = 1/2$, the curves of intersection of the surfaces corresponding to $\sigma' = 2,3,4,5,10$ and ∞ with a normal section θ = constant are indicated in fig. (7.2).

7.11 FLOWLINE IN THE CENTRAL PLANE

The differential equation for the stream line in the central plane is

$$\frac{dr}{U} = \frac{(l + r \sin\phi)}{W}\, d\theta \;, \; (\phi = \pm \pi/2) \tag{7.11.1}$$

which can be written as

$$\frac{d\theta}{dr} = \frac{W}{(l + r\sin\phi)\,V}, \qquad (\phi = \pm\tfrac{\pi}{2})$$

or

$$\frac{d\theta}{dr} = \frac{4\mu}{Aa^3}\cdot\frac{\tau}{\psi'}\cdot\frac{1}{\sin\phi}\cdot\Big[\frac{Aa^2}{4\mu}(1-\varsigma^2-\lambda\ln\varsigma)$$

$$+ \frac{Aa^3}{4\mu l}\cdot\omega'\sin\phi\Big],$$

$$(\phi = \pm\tfrac{\pi}{2}) \quad (7.11.2)$$

Equation (7.11.2) shows that the stream line in the central plane will depend on the ratio a/l. Following Dean (1928) to a sufficient approximation, we may write (7.11.2) as

$$\pm\frac{d\theta}{d\varsigma} = \frac{\varsigma(1-\varsigma^2-\lambda\ln\varsigma)}{\psi'} \qquad (7.11.3)$$

the sign $\pm$ being the sign of ϕ, and the value of ψ'-the one given by (7.4.4). Equation (7.11.3) shows that the flow lines will vary with σ and R. For $\sigma = 1/2$ we have to evaluate the integral

$$\pm\theta = \int \frac{\varsigma(1-\varsigma^2+1.082095\ln\varsigma)}{\psi'}\,d\varsigma \qquad (7.11.4)$$

where ψ' is given by (7.9.5).

The (θ,ς) relation given by (7.11.4) is seen to be independent of the ratio a/l. For a given value of ς, θ varies inversely as R. The magnitude of θ increases

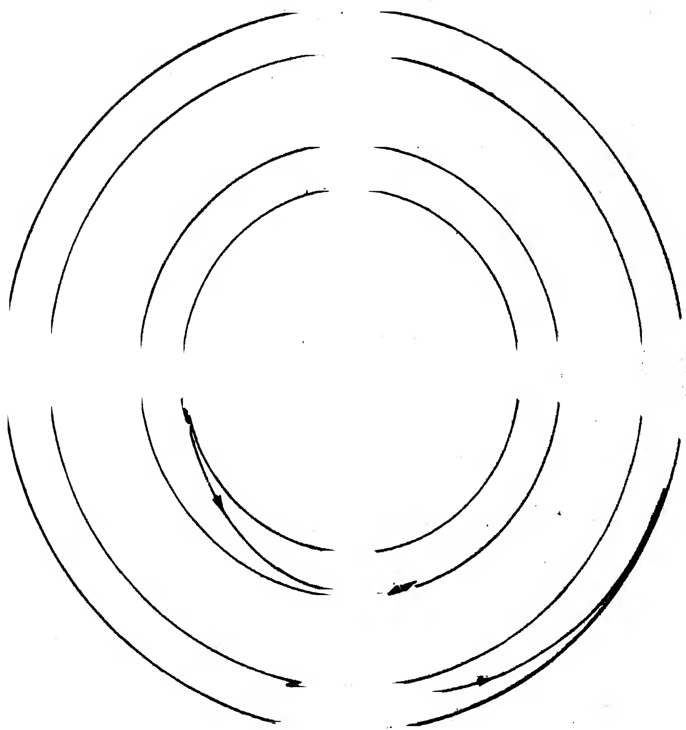

Figure 7.4 Flow-line in the central plane ($\phi = \pm\pi/2$)

continuously with ζ and tends to infinity as ζ tends to unity. Numerical illustrations are now given for = 100 and $\frac{a}{\ell} = \frac{1}{3}$ and for $\sigma = \frac{1}{3}$ (It is doubtful whether the approximations made in the above discussions are strictly valid for this choice of $\frac{a}{\ell}$). The integral (7.11.4) can be evaluated numerically to give θ as a function of ζ . The results of such a numerical integration are given in table 7.3.

Table 7.3 : Values of θ in degrees for corresponding values of ζ

Values of θ in degrees for corresponding value of ζ								
ζ	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
θ	8.94	16.60	23.01	28.65	33.43	38.65	46.16	56.81

The form of the stream line for the particular value $\sigma = \frac{1}{2}$ is shown in the figure (7.4).

7.12 THE EFFECT OF RADIUS RATIO ON VELOCITY FIELD

In order to examine the effect of the radii ratio $\sigma = \frac{b}{a}$ on the velocity field, we have performed the numerical computation on equations (7.5.2) and (7.3.10) - the perturbation velocity ω and the primary velocity ω_1 for the straight annulus. The results have been given in tabular form. Tables (7.5) to (7.10) give the numerical values of

$\left(\frac{4\mu}{Aa^2}\cdot \omega_1\right)$, $\left(\frac{4\mu}{Aa^2}\cdot \omega\right)$ and $\left(\frac{4\mu}{Aa^2}\cdot W\right)$ at various points ξ on the horizontal diameter of the annulus for fixed value of the Reynold's number $R = 200$ and for the curvature $\frac{a}{l} = .02$. The choice of R is arbitrary since it is within the limits of permissible Reynolds numbers since it is less than 2000 the critical number for stream line flow mentioned by Dean (1928) and also since ω- the secondary velocity is everywhere less than 5 percent of the primary velocity ω_1. This is justified for the first approximation taken for the problem. The radius ratio of the cylinders has been taken to vary from .05 to .1 at an interval of .01. The contribution of ω on the convex side ($\phi = \pi/2$) and the concave side ($\phi = -\frac{\pi}{2}$) of the horizontal diameter is just the reverse of each other i.e. on the concave side upto the point $\xi = .20$, the contribution of this velocity is positive and after that it becomes negative. On the convex side it has just the opposite effect. The point where the perturbation velocity ω changes sign move towards the outer wall as we increase the value of R and it reaches somewhere near $\xi = .35$ for $\sigma = .1$.

In tables (7.11) to '7.13) we have tabulated the velocity fields for fixed value of the radii ratio $\sigma = .025$ and $\frac{a}{l} = .05$ for different values of the Reynold number R at all points on the horizontal diameter. For $R = 50$, the contribution of ω is positive at all points of the horizontal

diameter on the concave side whereas it is negative on the convex side. It is also more prominent towards the inner wall than towards the outer wall. As R increases to $R = 75$, we see that upto $\zeta = .40$ the sign of ω is positive and then it becomes negative and again at $\zeta = .80$ and onwards it becomes positive. This shows that for some combinations of R and σ the secondary velocity may be zero. When R increases further to $R = 100$, the point of change of sign moves towards the inner wall.

Table 7.5 : The velocity field for R = 200, $\frac{a}{\ell}$ = .02 and σ = .05 .

	ζ	$\left(\frac{4\mu}{Aa^2} W_1\right)$	$\left(\frac{4\mu}{Aa^2} \cdot W\right)$	$\left(\frac{4\mu}{Aa^2} W\right)$
	.10	.223300	-.001416	.221884
	.15	.345809	-.001891	.343918
	.20	.424100	-.001883	[illegible]
	.25	.475900	+.003582	[illegible]
	.30	.509109	.002937	.512046
	.35	.527937	.006094	.534031
	.40	.534899	.009502	.544401
	.45	.531618	.012789	.544407
$\Phi = \frac{\pi}{2}$	.50	.319206	.015602	.534808
	.55	.498436	.017650	.516086
	.60	.469909	.018721	[illegible]
	.65	.434061	.018710	.452771
	.70	.391257	.017628	.408885
	.75	.341709	.015893	.357602
	.80	.285699	.012816	.298515
	.85	.223385	.009579	.232965
	.90	.154918	.006188	.161106
	.95	.080421	.002926	.083346

Table 7.6 : The velocity field for R = 200, $\frac{a}{l}$ = .02 and σ = .06 .

$\phi = \frac{\pi}{2}$

ζ	$\left(\frac{4\mu}{Aa^2} w_1\right)$	$\left(\frac{4\mu}{Aa^2} \cdot w\right)$	$\left(\frac{4\mu}{Aa^2} W\right)$
.10	.174514	-.000245	.173839
.15	.305614	-.001400	.304215
.20	.390000	-.001203	.388797
.25	.446529	-.000166	.446363
.30	.483600	+.001646	.485246
.35	.505694	.004043	.509737
.40	.515486	.006754	.522239
.45	.514700	.009468	.524167
.50	.504514	.011875	.516389
.55	.485770	.013704	.499473
.60	.459086	.014760	.473846
.65	.424934	.014901	.439835
.70	.383680	.014144	.397824
.75	.335614	.012565	.348179
.80	.280971	.010342	.291313
.85	.219942	.007716	.227658
.90	.182985	.004959	.187944
.95	.079334	.002294	.081628

Table 7.7 : The velocity field for R = 200, $\frac{a}{\ell}$ = .02
and σ = .07

$\Phi = \frac{\pi}{2}$

y	$\left(\frac{4\mu}{Aa^2} w_1\right)$	$\left(\frac{4\mu}{Aa^2} w\right)$	$\left(\frac{4\mu}{Aa^2} W\right)$
.10	.128368	-.000462	.127906
.15	.267594	-.000994	.266600
.20	.357745	-.001040	.356705
.25	.418746	-.000446	.418300
.30	.459471	+.000302	.459773
.35	.484655	.002596	.487251
.40	.497122	.004735	.501857
.45	.498997	.006962	.505959
.50	.490623	.009008	.499631
.55	.473788	.010626	.484414
.60	.448848	.011621	.460469
.65	.416300	.011874	.428174
.70	.376532	.011356	.387888
.75	.329849	.010131	.339980
.80	.276499	.008346	.284846
.85	.216995	.006210	.223205
.90	.150574	.003964	.154537
.95	.078306	.001836	.080142

Table 7.8 : The velocity field for R = 200, $\frac{a}{\ell}$ = .02 and σ = .08 .

	ξ	$\left(\frac{4\mu}{Aa^2}\cdot w_1\right)$	$\left(\frac{4\mu}{Aa^2}\cdot w\right)$	$\left(\frac{4\mu}{Aa^2}W\right)$
$\phi = \frac{\pi}{2}$	.10	.084183	-.000221	.083962
	.15	.231189	-.000584	.230605
	.20	.326061	-.000364	.325697
	.25	.392144	-.000580	.391564
	.30	.436367	+.000241	.436608
	.35	.464509	.001555	.466064
	.40	.479539	.003221	.482760
	.45	.483374	.006033	.489407
	.50	.477322	.006760	.484082
	.55	.462316	.008179	.470495
	.60	.439046	.009105	.448151
	.65	.408034	.009420	.417454
	.70	.369087	.009683	.378770
	.75	.321328	.008138	.329466
	.80	.272217	.006706	.278923
	.85	.213566	.004970	.218536
	.90	.148552	.003143	.151695
	.95	.077322	.001434	.078756

Table 7.9 : The velocity field for R = 209, $\frac{a}{l}$ = .02
and σ = .09 .

	ζ	$\left(\frac{4\mu}{Aa^2}\cdot w_1\right)$	$\left(\frac{4\mu}{Aa^2}\cdot \omega\right)$	$\left(\frac{4\mu}{Aa^2}\cdot W\right)$
	.10	.041501	-.000079	.041422
	.15	.196023	-.000459	.195564
	.20	.297028	-.000704	.296324
	.25	.366447	-.000632	.365815
	.30	.414050	-.000134	.413916
	.35	.445049	+.000797	.445846
	.40	.462554	.002071	.464625
	.45	.468572	.003529	.472101
	.50	.464473	.004973	.469447
$\Phi = \frac{\pi}{2}$	.55	.451234	.006206	.457440
	.60	.429577	.007056	.436633
	.65	.400048	.007404	.407452
	.70	.363076	.007204	.370279
	.75	.318996	.006481	.325477
	.80	.268081	.005338	.273419
	.85	.210554	.003933	.214487
	.90	.145609	.003457	.149066
	.95	.076371	.001097	.077468

Table 7.10 : The velocity field for $R = 200$, $\frac{a}{\ell} = .02$ and $\sigma = .1$

	ξ	$\left(\frac{4\mu}{Aa^2}\cdot w_1\right)$	$\left(\frac{4\mu}{Aa^2}\cdot \omega\right)$	$\left(\frac{4\mu}{Aa^2} W\right)$
	.10	.000000	.000000	.000000
	.15	.161830	-.000304	.161526
	.20	.268020	-.000571	.267449
	.25	.341461	-.000642	.340819
	.30	.392350	-.000386	.391964
	.35	.426127	+.000239	.426367
	.40	.446039	.001189	.447228
	.45	.454180	.002943	[illegible]
	.50	.451980	.003539	.455519
$\varphi = \frac{\pi}{2}$	.55	.440459	.004600	.445059
	.60	.420370	.005369	.425739
	.65	.392284	.005732	.398016
	.70	.356647	[illegible]	.362221
	.75	.313811	.005092	[illegible]
	.80	.264059	.004187	.268246
	.85	.207625	.003059	.210684
	.90	.144700	.001877	.146577
	.95	.075446	.000813	.076259

Table 7.11 : The velocity field for R = 50, $\frac{a}{\ell}$ = .05
and σ = .025 .

	ξ	$\left(\frac{4\mu}{Aa^2}\omega_1\right)$	$\left(\frac{4\mu}{Aa^2}\cdot\omega\right)$	$\left(\frac{4\mu}{Aa^2}\cdot W\right)$
	.25	.561931	-.005241	.556690
	.30	.583825	-.005610	.578215
	.35	.593087	-.005807	.587280
	.40	.591763	-.005894	.585869
	.45	.581172	-.005924	.575248
	.50	.562216	-.005939	.556277
	.55	.535537	-.005960	.529577
	.60	.501609	-.005989	.495620
$\phi = \frac{\pi}{2}$	.65	.460794	-.006005	.454789
	.70	.413371	-.005962	.407409
	.75	.359562	-.005794	.353768
	.80	.299547	-.005418	.294129
	.85	.233471	-.004741	.228730
	.90	.161456	-.003667	.157789
	.95	.083604	-.002110	.081494

Table 7.12 : The velocity field for $R = 75$, $\frac{a}{l} = .05$ and $\sigma = .025$.

	ζ	$\left(\frac{4\mu}{Aa^2} \cdot w_1\right)$	$\left(\frac{4\mu}{Aa^2} \cdot w\right)$	$\left(\frac{4\mu}{Aa^2} \cdot W\right)$
	.25	.561931	-.003777	.558155
	.30	.583825	-.002912	.580913
	.35	.593087	-.001806	.591282
	.40	.591763	-.000635	.591128
	.45	.581172	+.000433	.581605
	.50	.562216	+.001268	.5[illegible]483
	.55	.535637	+.001777	.537313
$\phi = \frac{\pi}{2}$	.60	.501609	+.001919	.50[illegible]
	.65	.460794	+.001705	.4[illegible]499
	.70	.413371	+.001199	.41[illegible]570
	.75	.359562	+.000513	.360075
	.80	.299547	-.000205	.29[illegible]41
	.85	.233471	-.000783	.232[illegible]
	.90	.161456	-.001041	.160415
	.95	.083604	-.000820	.082784

Table 7.13 : Velocity field for $R = 100$, $\frac{a}{l} = .05$ and $\sigma = .025$

	ρ	$\left(\frac{4\mu}{Aa^2}\cdot w_1\right)$	$\left(\frac{4\mu}{Aa^2}\cdot w\right)$	$\left(\frac{4\mu}{Aa^2}\cdot W\right)$
$\phi = \frac{\pi}{2}$	.25	.561931	-.001727	[illegible]
	.30	.583825	.000864	.584689
	.35	.593087	.003798	[illegible]
	.40	.591763	.006722	[illegible]
	.45	.581172	.009334	.590506
	.50	.562216	.011356	.573572
	.55	.535537	.012608	.548145
	.60	.501609	.012990	.514600
	.65	.460794	.012499	.473293
	.70	.413371	.011225	.424596
	.75	.352662	.009342	.362004
	.80	.299647	.007092	.306639
	.85	.233471	.004759	[illegible]
	.90	.161456	.002636	.164092
	.95	.082604	+.000986	[illegible]

CHAPTER VIII

CONVECTIVE HEAT-TRANSFER IN A CURVED PIPE STREAMLINE FLOW

8.1 INTRODUCTION

Consider a laminar and steady flow of a Newtonian fluid through a curved pipe whose cross-section is a simply connected circular region and the curve joining the centres of cross-sections is a circular arc. Let the wall temperature be constant with respect to time as well as direction, fluid properties (i.e. viscosity coefficient, thermal conductivity coefficient etc.) are constant, free convection effects are negligible (i.e. flow is a forced flow), heat radiation and heat source (or sink) distributions (such as : chemical or nuclear etc.) are absent and both velocity and temperature profiles are fully developed. We confine ourselves to the

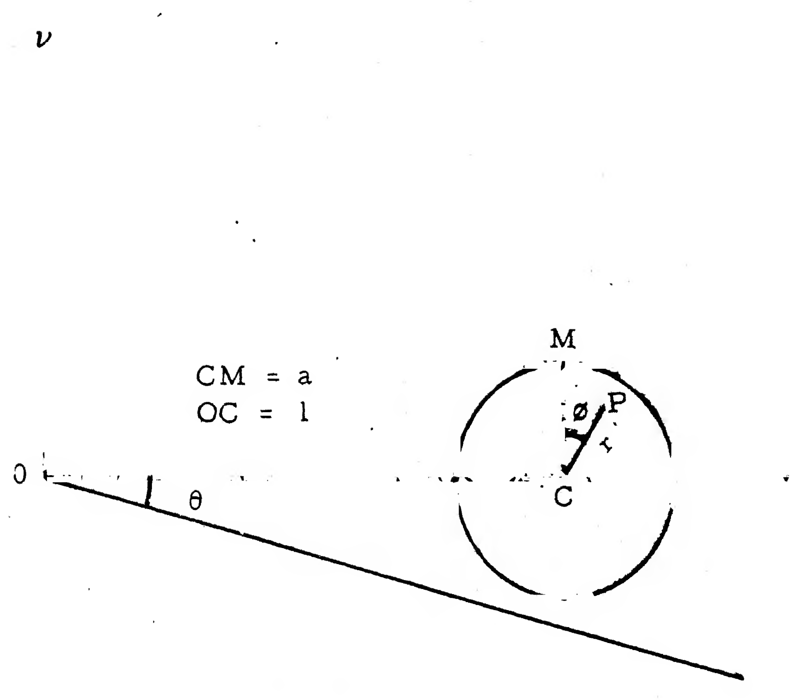

Figure 8.1 The coordinate system (r, ϕ, θ) chosen to describe motion in a curved pipe of circular cross-section.

case of liquids only so that we can omit heat of compression. Thus, dissipation function is the only representative of heat transfer in the problem. The problem for straight duct was first of all studied by Schlichting (1951). The solution of the corresponding problem of straight pipe of circular cross-section can be seen in Landau and Lifshitz (1959). The purpose of this chapter is to extend the work of Dean (1927) to include the case of heat generation due to viscous dissipation. We follow Dean and evaluate the temperature profile at the same level of dissipation at which the velocity profile has been evaluated by Dean.

8.2 GENERAL EQUATIONS AND TEMPERATURE FIELD

A circular pipe of radius a is coiled in a circle of radius ℓ, i.e. the locus of the centres of all cross-sections makes a circle of radius ℓ. The coordinate system used by us is the same as was adopted by Dean (1927) and Jones (1960) in discussing the curved pipe flow problem and is represented in the figure (8.1). $O\nu$ is the axis of the circle in which the pipe is coiled. C is the centre of the circular cross-section by a plane which passes through $O\nu$ and makes an angle θ with a fixed plane $(\theta = 0)$ passing through $O\nu$. CO is perpendicular to $O\nu$ and of length ℓ. The plane through O perpendicular to $O\nu$ will be called the 'central plane' of the pipe and the circle traced out by C its 'central line'. Any point P of the cross-section

is referred to by orthogonal coordinates r, ϕ, θ where r is the distance CP and ϕ is the angle made by CP with the line through C parallel to OV. The surface of the pipe is given by $r=a$, a being the radius of any section. The components of velocity corresponding to these coordinates are (U, V, W); U is therefore in the direction CP, V perpendicular to U and in the plane of the cross-section, and W perpendicular to this plane. The general direction of flow will be taken to be in the direction in which θ increases.

When the incompressible Newtonian fluid characterised by the rheological equation of state

$$t_{ij} = -p\,\delta_{ij} + 2\mu\, e_{ij}, \tag{8.2.1}$$

flows through the curved pipe (due to the fall in pressure along the pipe) with a uniform wall temperature, the temperature profile is changed from the uniform profile at the entrance to the characteristic one down stream and at a later instant a fully developed temperature profile is formed inside the pipe. To this temperature is added the heat of viscous dissipation. The viscous dissipation function for the above liquid is given by

$$\begin{aligned}\Phi &= 2\mu\, e_{ij}\, e_{ij} \\ &= 2\mu\left(e_{rr}^2 + e_{\phi\phi}^2 + e_{\theta\theta}^2 + 2e_{r\phi}^2 + 2e_{\theta\phi}^2 + 2e_{r\theta}^2\right)\end{aligned} \tag{8.2.2}$$

The general energy equation, for the liquids of constant properties with zero internal heat source and zero radiation is

$$\rho C_p \frac{DT}{Dt'} = -\frac{T}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p \frac{Dp}{Dt'} + \Phi + K \nabla^2 T, \tag{8.2.3}$$

where K is the thermal conductivity, ρ is the density and C_p is the specific heat at constant pressure. The quantity $-\frac{T}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p$ is usually very small for liquids. It is, therefore, taken equal to zero in the present case. The equation of continuity is

$$\frac{\partial U}{\partial r} + \frac{U}{r} + \frac{U \sin\phi}{l + r\sin\phi} + \frac{1}{r}\frac{\partial V}{\partial \phi} + \frac{V\cos\phi}{l + r\sin\phi} = 0 \tag{8.2.4}$$

Following Dean (1927) we assume

$$U = u, \quad V = v, \quad W = \frac{A}{4\mu}(a^2 - r^2) + w$$

and

$$P = -Az + B + p$$

where u, v, w and p are referred to the secondary flow and all are taken to be small of the order of $\frac{a}{l}$ and u, v, w are independant of θ . We further rewrite

$$w_1 = \frac{A}{4\mu}(a^2 - r^2) \tag{8.2.5}$$

$$p_1 = -Az + B$$

which are nothing but the velocity and the pressure fields for the flow in a straight circular pipe under the constant pressure gradient. Thus in this case we can regard u, v, w and p as small. With this the equation of continuity takes the form

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \phi} = 0 \tag{8.2.6}$$

and this suggests the form for the stream function ψ as

$$u = -\frac{1}{r}\frac{\partial \psi}{\partial \phi} \qquad \text{and} \qquad v = \frac{\partial \psi}{\partial r}. \tag{8.2.7}$$

Thus the strain-rate and the stress components are simplified and the simplified equations of motion reduce to

$$-\rho \frac{A^2}{16\mu^2 l}(a^2-r^2)^2 \sin\phi = -\frac{\partial p}{\partial r} - \frac{\mu}{r}\frac{\partial}{\partial \phi}\left(\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \phi^2}\right) \tag{8.2.8}$$

$$-\rho \frac{A^2}{16\mu^2 l}(a^2-r^2)^2 \cos\phi = -\frac{1}{r}\frac{\partial p}{\partial \phi} + \mu\frac{\partial}{\partial r}\left(\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \phi^2}\right) \tag{8.2.9}$$

$$\rho\frac{A}{2\mu}\frac{\partial \psi}{\partial \phi} + \frac{3A}{2l} r \sin\phi = \mu\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \phi^2}\right) \tag{8.2.10}$$

where $\frac{\partial p}{\partial z}$ was found to be constant and has been taken to be zero for simplicity.

8.3 ENERGY EQUATIONS

For the straight circular pipe, we have

$$\rho C_p \frac{D t_{1\ell}}{D t'} = K \overset{2}{\Delta} t_{1\ell} + \Phi \tag{8.3.1}$$

$$0 = K\left(\frac{d^2 t_{1\ell}}{d r^2} + \frac{1}{r}\frac{d t_{1\ell}}{d r} \right) \tag{8.3.2}$$

and

$$t_{1\ell} = t_\omega + \frac{A^2}{64 \mu K}\left(a^4 - r^4 \right) \tag{8.3.3}$$

where t_ω is the constant wall temperature, and $t_{1\ell}$ is the temperature at a point inside the straight pipe. Just as in the case of velocity field in the curved pipe (see Dean (1927)), we write down the temperature field as a perturbation t_ℓ over the straight pipe case temperature and this perturbation is small due to the large radius of curvature ℓ . Let

$$T_\ell(r,\phi) = t_{1\ell}(r,\phi) + t_\ell(r,\phi)$$

or simply (8.3.4)

$$T_\ell = t_{1\ell} + t_\ell$$

Here t_ℓ occuring in (8.3.4) has been written at the level of approximation at which the velocity components and pressure have been written in Dean (1927).

For liquids, the energy equation (8.2.3) is written as

$$\rho C_p u \frac{d t_{1\ell}}{dr} = 2\mu \frac{d w_1}{dr}\left(\frac{dw}{dr} - \frac{w_1 \sin\phi}{\ell}\right) +$$

$$K \frac{\sin\phi}{\ell} \frac{d t_{1\ell}}{dr} + K\left(\frac{\partial^2 t_\ell}{\partial r^2} + \frac{1}{r}\frac{\partial t_\ell}{\partial r} + \frac{1}{r^2}\frac{\partial^2 t_\ell}{\partial \phi^2}\right) \qquad (8.3.5)$$

Since $\frac{d t_{1\ell}}{dr} = -\frac{A^2 r^3}{16\mu K}$ and $\frac{d w_1}{dr} = -\frac{A r}{2\mu}$ (8.3.6)

Equation (8.3.5) simplifies to

$$\rho C_p \frac{A^2 r^2}{16\mu K}\frac{\partial \psi}{\partial \phi} + A r \frac{\partial w}{\partial r} - \frac{A^2 r}{16\mu\ell}(4a^2 - 5r^2)\sin\phi$$

$$= K\left(\frac{\partial^2 t_\ell}{\partial r^2} + \frac{1}{r}\frac{\partial t_\ell}{\partial r} + \frac{1}{r^2}\frac{\partial^2 t_\ell}{\partial \phi^2}\right) \qquad (8.3.7)$$

8.4 SOLUTIONS

Following Dean (1927) and Jones (1960), the equations of motion (8.2.8) to (8.2.10) can be solved by assuming ψ, w and p of the form

$$\psi = \frac{A a^4}{4\mu\ell}\psi'(\xi)\cos\phi$$

$$w = \frac{A a^3}{4\mu\ell} w'(\xi)\sin\phi \qquad (8.4.1)$$

and

$$p = \frac{A a^2}{4\ell} p'(\xi)\sin\phi$$

where $\psi'(\xi)$, $w'(\xi)$ and $p'(\xi)$ are dimensionless functions of $\xi = \frac{r}{a}$. Thus these equations, when substituted in the equations of motion (8.2.8) to (8.2.10) reduce to

$$-R(1-\xi)^2 = -\frac{dp'}{d\xi} + \frac{1}{\xi}\Delta\psi' \tag{8.4.2}$$

$$-R(1-\xi)^2 = -\frac{p'}{\xi} + \frac{d}{d\xi}\Delta\psi' \tag{8.4.3}$$

$$-2R\psi' + 6\xi = \Delta w' \tag{8.4.4}$$

where

$$R = \frac{\rho A a^3}{4\mu^2} \quad \text{and} \quad P_r = \frac{\mu C_p}{K}$$

and the operator is defined by

$$\Delta \equiv \frac{\partial^2}{\partial\xi^2} + \frac{1}{\xi}\frac{\partial}{\partial\xi} - \frac{1}{\xi^2}$$

Consistent with the form of ψ, w and P the solution for t_ℓ will be of the form

$$t_\ell = \frac{A^2 a^5}{64\mu K \ell} t'_\ell(\xi) \sin\phi \tag{8.4.5}$$

Substituting the values of $\frac{\partial\psi}{\partial\phi}$ and $\frac{\partial w}{\partial r}$ etc. in (8.3.8) we get

$$-4P_r R\xi^2\psi' - 4\xi(4-5\xi^2) + 16\xi\frac{\partial w'}{\partial\xi} = \Delta t'_\ell \tag{8.4.6}$$

Given this equation we can determine t_e' provided $\frac{\partial \omega'}{\partial \xi}$ and ψ' are known and these we have from Jones and Dean as

$$\psi' = \frac{R}{288}\left(4\xi - 9\xi^3 + 6\xi^5 - \xi^7\right) \tag{8.4.7}$$

$$\omega' = -\frac{3}{4}\left(\xi - \xi^3\right) + \frac{R^2}{11520}\left(19\xi - 40\xi^3 + 30\xi^5 - 10\xi^7 + \xi^9\right) \tag{8.4.8}$$

and

$$p' = \frac{R}{12}\left(19\xi - 6\xi^3 + 2\xi^5\right). \tag{8.4.9}$$

Thus

$$\begin{aligned} \Delta t_e' &= \frac{d}{d\xi}\left\{\frac{1}{\xi}\frac{d}{d\xi}\left(\xi t_e'\right)\right\} \\ &= -\frac{P_r R^2}{72}\left(4\xi^3 - 9\xi^5 + 6\xi^7 - \xi^9\right) - 4\left(4\xi - 5\xi^3\right) \\ &\quad -12\left(\xi - 3\xi^3\right) + \frac{R^2}{720}\left(19\xi - 120\xi^3 + 150\xi^5 - 70\xi^7 + 9\xi^9\right) \end{aligned} \tag{8.4.10}$$

and therefore

$$\begin{aligned} \frac{d}{d\xi}\left(\xi t_e'\right) &= -\frac{R^2 P_r \xi^5}{1440}\left(20 - 30\xi^2 + 15\xi^4 - 2\xi^6\right) - 14\xi^3\left(1 - \xi^2\right) \\ &\quad + \frac{R^2}{14400}\,\xi^3\left(190 - 600\xi^2 + 500\xi^4 - 175\xi^6 + 18\xi^8\right) + B_1\xi \end{aligned} \tag{8.4.11}$$

B_1 being arbitrary constant of integration. This equation again on integration gives

$$t_l' = -\frac{7}{6}\xi^3(3-2\xi^2) - \frac{R^2 P_r \xi^5}{17280}(40-45\xi^2+18\xi^4-2\xi^6)$$

$$+\frac{R^2\xi^3}{28800}(95-200\xi^2+125\xi^4-35\xi^6+3\xi^8)+\frac{B_1\xi}{2}+\frac{B_2}{\xi}, \tag{8.4.12}$$

where B_2 is again an arbitrary constant of integration. If this constant B_2 were finite t_l' and hence the temperature would be infinite at the centre of the cross-section. Since the temperature at any point inside the pipe has to be finite and not infinite, the constant B_2 must be zero. In order to determine B_1, we have the thermal boundary conditions $t_l = t_l' = 0$ at $\xi = 1$ from the prescribed condition that $T_l = t_w$ at $\xi = 1$ and therefore from equations (8.3.3) and (8.3.4) we get

$$B_1 = \frac{7}{3} + \frac{11 R^2 P_r}{8640} + \frac{R^2}{1200}. \tag{8.4.13}$$

and therefore, we get

$$t_l' = \left(\frac{7}{6} + \frac{11 R^2 P_r}{17280} + \frac{R^2}{2400}\right)\xi - \frac{7}{6}\xi^3(3-2\xi^2)$$

$$-\frac{R^2 P_r}{17280}\xi^5(40-45\xi^2+18\xi^4-2\xi^6)$$

$$+\frac{R^2}{28800}\xi^3(95-200\xi^2+125\xi^4-35\xi^6+3\xi^8). \tag{8.4.14}$$

From equation (8.4.14), we note that $t'_\ell = 0$ when $\xi = 0$. As a matter of fact t'_ℓ should vanish at the centre of the cross-section, otherwise the solution will not be of any physical significance. This is explained as follows: Consider two points on OC, one on the left hand and the other on the right hand sides of C. At the left hand side point $\phi = -\frac{\pi}{2}$ and $\sin\phi = -1$, while at the other $\phi = \frac{\pi}{2}$ and $\sin\phi = 1$. There will therefore be a finite difference in the values of t'_ℓ (and hence, in the temperatures also) at these points, however closer the points may be unless $t'_\ell = 0$ at $\xi = 0$. This implies that the temperature is undefined or discontinuous at $\xi = 0$ when $t'_\ell \neq 0$ at $\xi = 0$. This is absurd from the point of view of physics. Thus, for the solutions to be of any physical significance, t'_ℓ must necessarily vanish at $\xi = 0$.

Thus the temperature field at any point inside the pipe is given by

$$
\begin{aligned}
T_\ell &= t_{1\ell} + t_\ell \\
&= t_{1\ell} + \frac{A^2 a^5}{64\mu K \ell} \sin\phi \, t'_\ell \\
&= t_\omega + \frac{A^2 a^4}{64\mu K}(1-\xi^4) + \frac{A^2 a^5}{64\mu K\ell}\sin\phi \Big[\xi\,(20160 + \\
&\quad 11R^2 P_r + 7.2R^2)/17280 - \frac{7}{6}\xi^3(3-2\xi^2) \\
&\quad - \frac{R^2 P_r \xi^5}{17280}(40 - 45\xi^2 + 18\xi^4 - 2\xi^6) \qquad \textbf{(8.4.15)} \\
&\quad + \frac{R^2}{28800}\xi^3(95 - 200\xi^2 + 125\xi^4 - 35\xi^6 + 3\xi^8)\Big]
\end{aligned}
$$

or

$$\frac{T_\ell - t_\omega}{A^2 a^4/64\mu K} = (1-\xi^4) + \frac{a}{\ell}\sin\phi\left[\frac{7}{6}(\xi - 3\xi^3 + 2\xi^5) - \right.$$

$$\frac{R^2 P_r}{17280}(-11\xi + 40\xi^5 - 45\xi^7 + 18\xi^9 - 2\xi^{11})$$

$$\left. + \frac{R^2}{28800}(12\xi + 95\xi^3 - 200\xi^5 + 125\xi^7 - 35\xi^9 + 3\xi^{11})\right] \tag{8.4.17}$$

The rate of heat-transfer $\frac{\partial T_\ell}{\partial \xi}$ is given by

$$\frac{64\mu K}{A^2 a^4}\frac{\partial T_\ell}{\partial \xi} = -4\xi^3 + \frac{a}{\ell}\sin\phi\left[\frac{7}{6}(1 - 9\xi^2 + 10\xi^4) - \right.$$

$$\frac{R^2 P_r}{17280}(-11 + 20\xi^4 - 315\xi^6 + 162\xi^8 - 22\xi^{10})$$

$$\left. + \frac{R^2}{28800}(12 + 285\xi^2 - 1000\xi^4 + 875\xi^6 - 315\xi^8 + 33\xi^{10})\right] \tag{8.4.18}$$

and its value at the surface of the pipe i.e. $\xi = 1$ is given by

$$\frac{64\mu K}{A^2 a^4}\left.\frac{\partial T_\ell}{\partial \xi}\right|_{\xi=1} = -4 + \frac{a}{\ell}\sin\phi\left[\frac{7}{3} + \frac{266 R^2 P_r}{17280} - \frac{11 R^2}{2880}\right]. \tag{8.4.19}$$

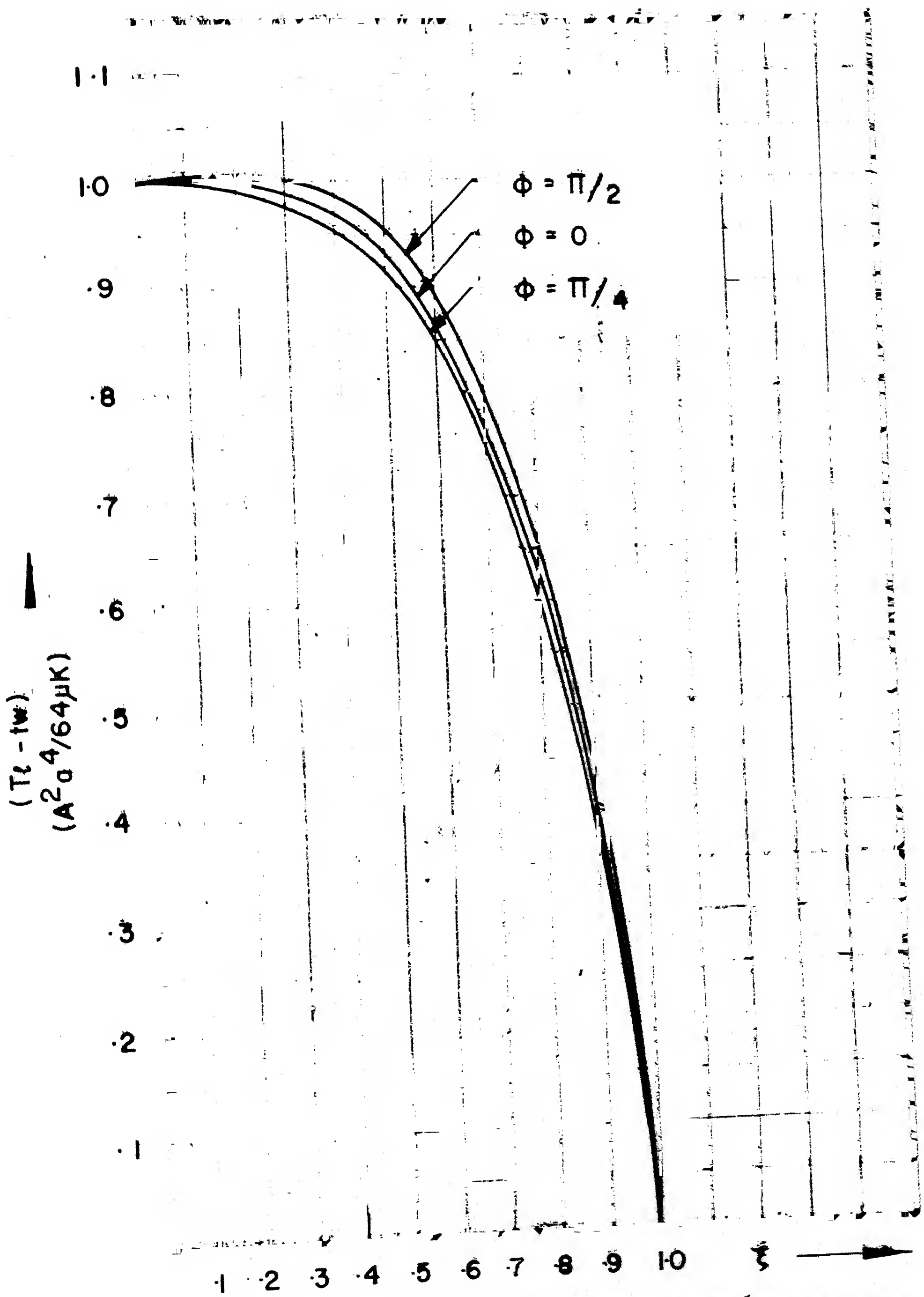

Figure 8·2- The temperature field inside the curved pipe for $Pr = 1·0, R = 50$ on all the radii vectors.

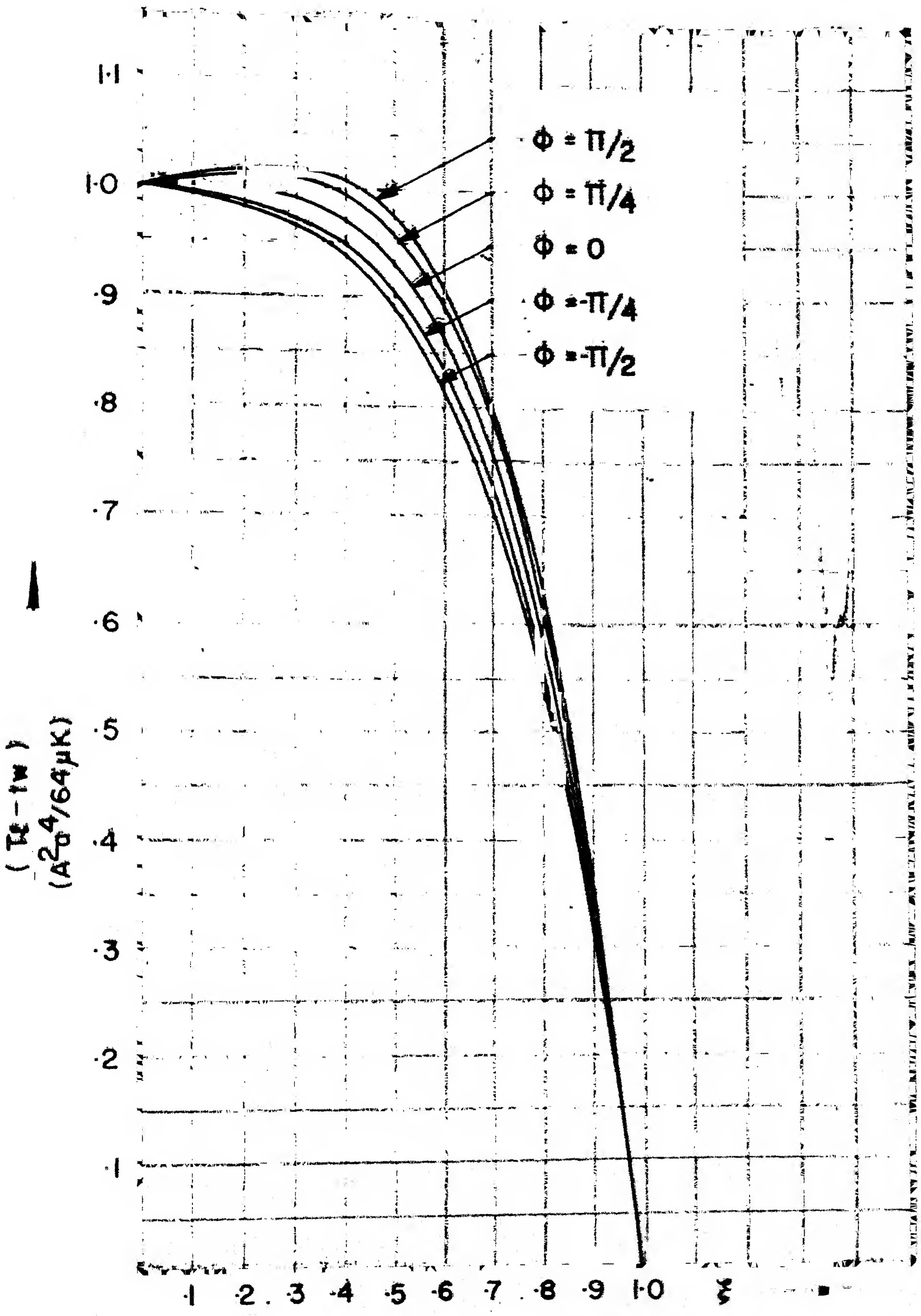

Figure 8·3 - The temperature field inside the curved pipe for Pr = 1·0 ; R = 75 on all the radii vectors.

8.5 DISCUSSIONS

We have carried out numerical evaluation of the temperature field and the local heat transfer at the wall of the pipe on IBM 1620 Computer. The Prandtl number P_r has been taken to be 1 or 7.5 . The Reynold's numbers R has been taken to be 50 or 75 or 100 along with the ratio of the radius of the cross-section of the pipe to that of the radius of curvature of the coil to be .01. We have drawn a number of graphs showing the behaviour of the temperature curves and the local heat-transfer rate curves for various combinations of R and P_r . From the graphs we may draw the following conclusions.

(i) Graph (8.2), shows the temperature field inside the pipe for P_r =1.0 and R = 50 on all the radii vectors. We notice that the temperature inside the pipe on the convex half of the pipe is a little higher than the corresponding points on the concave half of the pipe. On the horizontal diameter on the concave side at about ξ = .2 the temperature attains maximum value which is above the wall temperature.

(ii) Graph (8.3) has been drawn for R = 75 keeping P_r the same as 1.0 . Behaviour of the temperature curve remains the same but for a little change in temperature towards higher side. The point ξ = .2 also has a tendency to shift outwards. At no point on the concave side, the temperature inside pipe is above the temperature of the wall of the pipe.

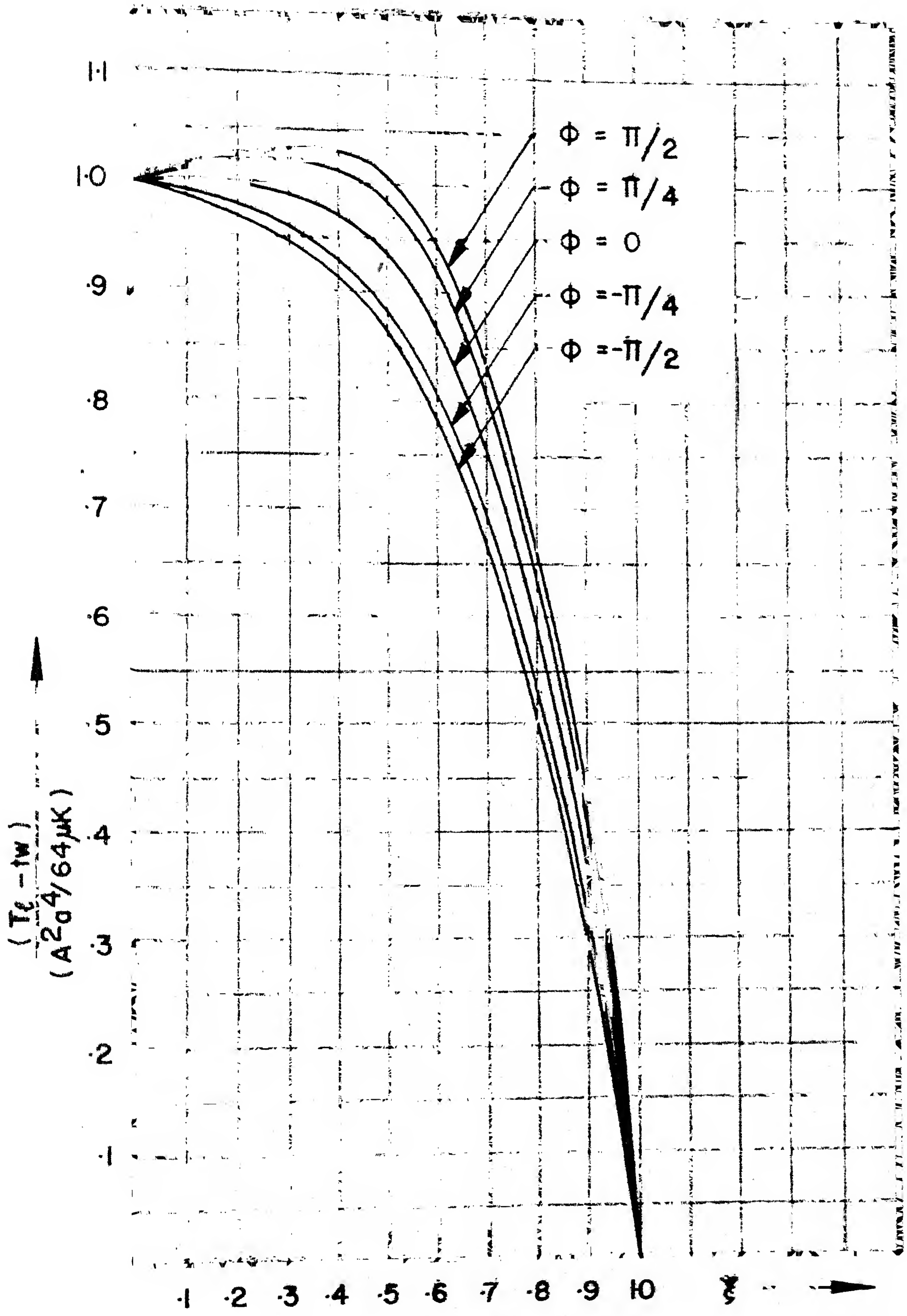

Figure 8·4 - The temperature field inside the curved pipe for Pr = 7·5, R = 50 on all the radii vectors.

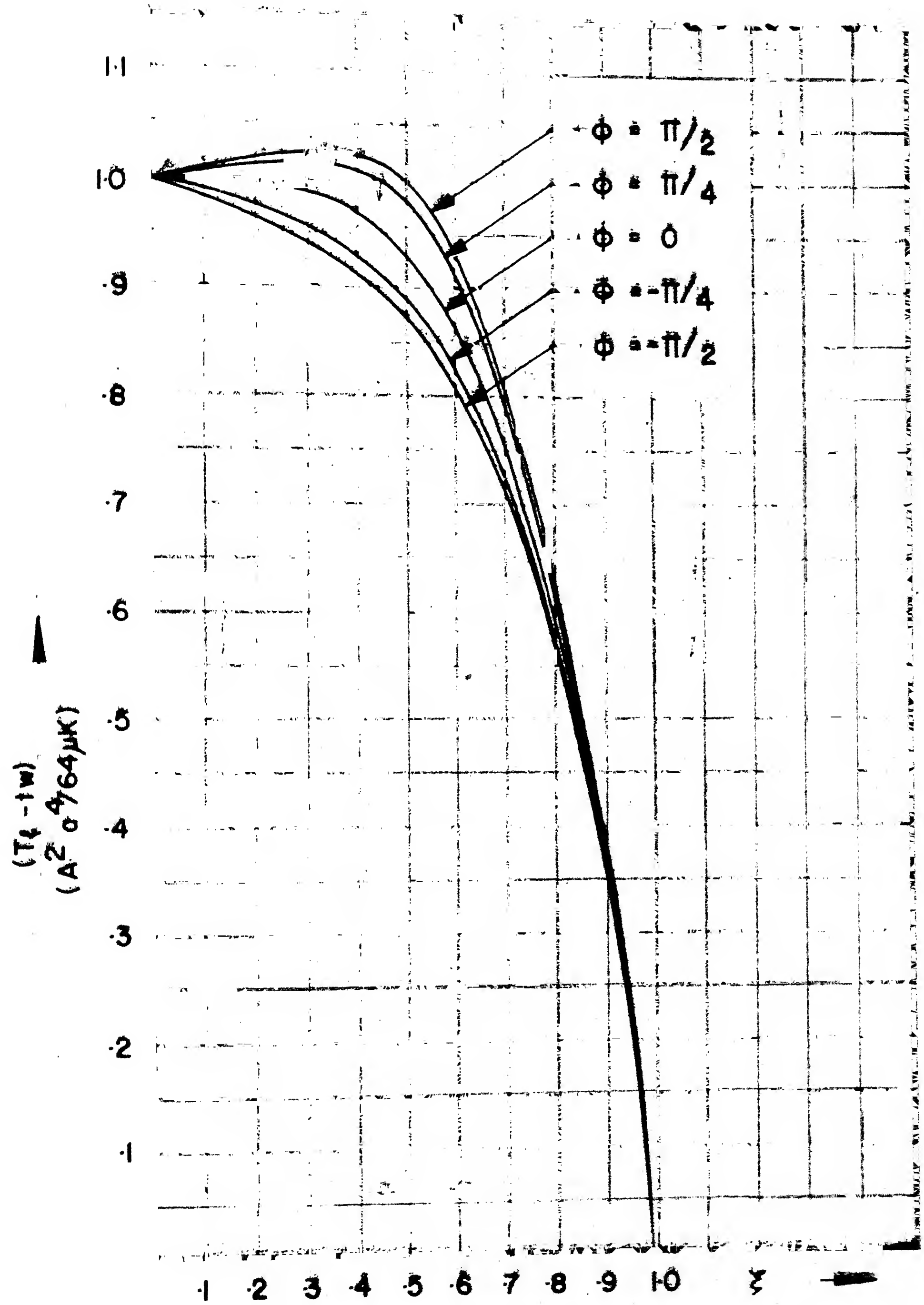

Figure 8·5 – The temperature field inside the curved pipe for Pr = 7·5, R = 50 on all the radii vectors.

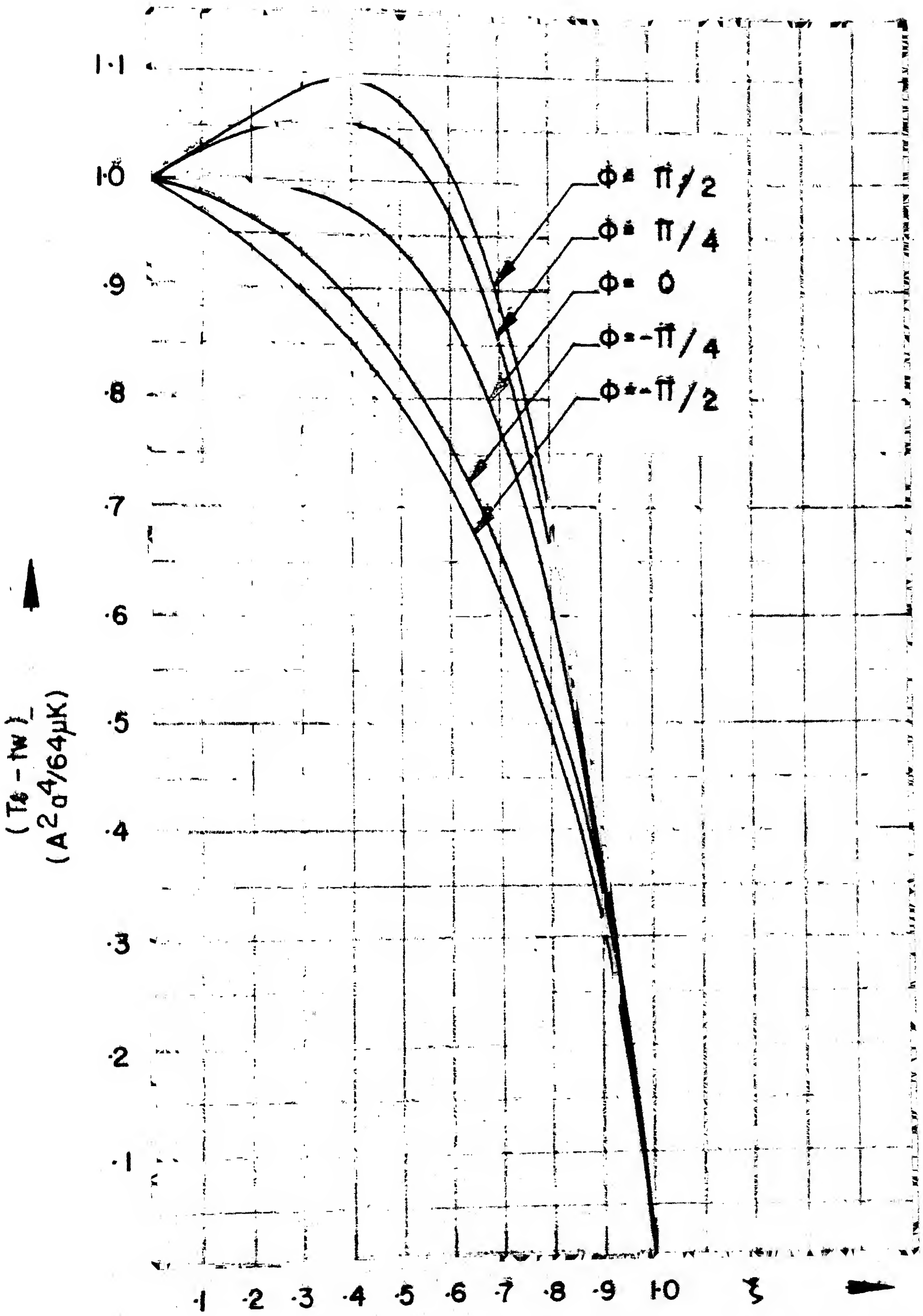

Figure 8·6–The temperature field inside the curved pipe for Pr = 75, R = 75 on all the radii vectors.

(iii) Graph (8.4) shows the temperature curve for R = 100 and P_r remaining the same at 1.0 . Here also the behaviour of the temperature field remains exhibiting a more appreciable difference in temperature on various radii vectors. At corresponding points on the convex and the concave side of the pipe, the temperature difference goes on increasing till ξ = .6 and then goes on decreasing and becomes the same at ξ = 1.00 as in every other case.

(iv) Graph (8.5) shows the temperature curves R = 50 and P_r = 7.5 to illustrate the effect of increase of P_r . Behaviour of these curves is again the same, and there is appreciable increase in the temperature at a point for the corresponding curve when R = 50 and P_r = 1.0 . On $\phi = \frac{\pi}{2}$ and $\phi = -\frac{\pi}{2}$ the temperature difference is maximum at about ξ = .6 .

(v) Graph (8.6) shows the temperature curves for R = 75 and P_r = 7.5 on horizontal diameter, the vertical diameter and the radii vectors $\phi = \frac{\pi}{4}$ and $\phi = -\frac{\pi}{4}$. The temperature on the vertical diameter remains the same as in all the other cases, and this goes on increasing with increase in R , such that on the horizontal diameter on the convex side of the pipe, it attains maximum value at about ξ = .4 . The maximum temperature difference is observed to be on the points ξ = .6 on horizontal diameter on opposite sides of the centre of the pipe.

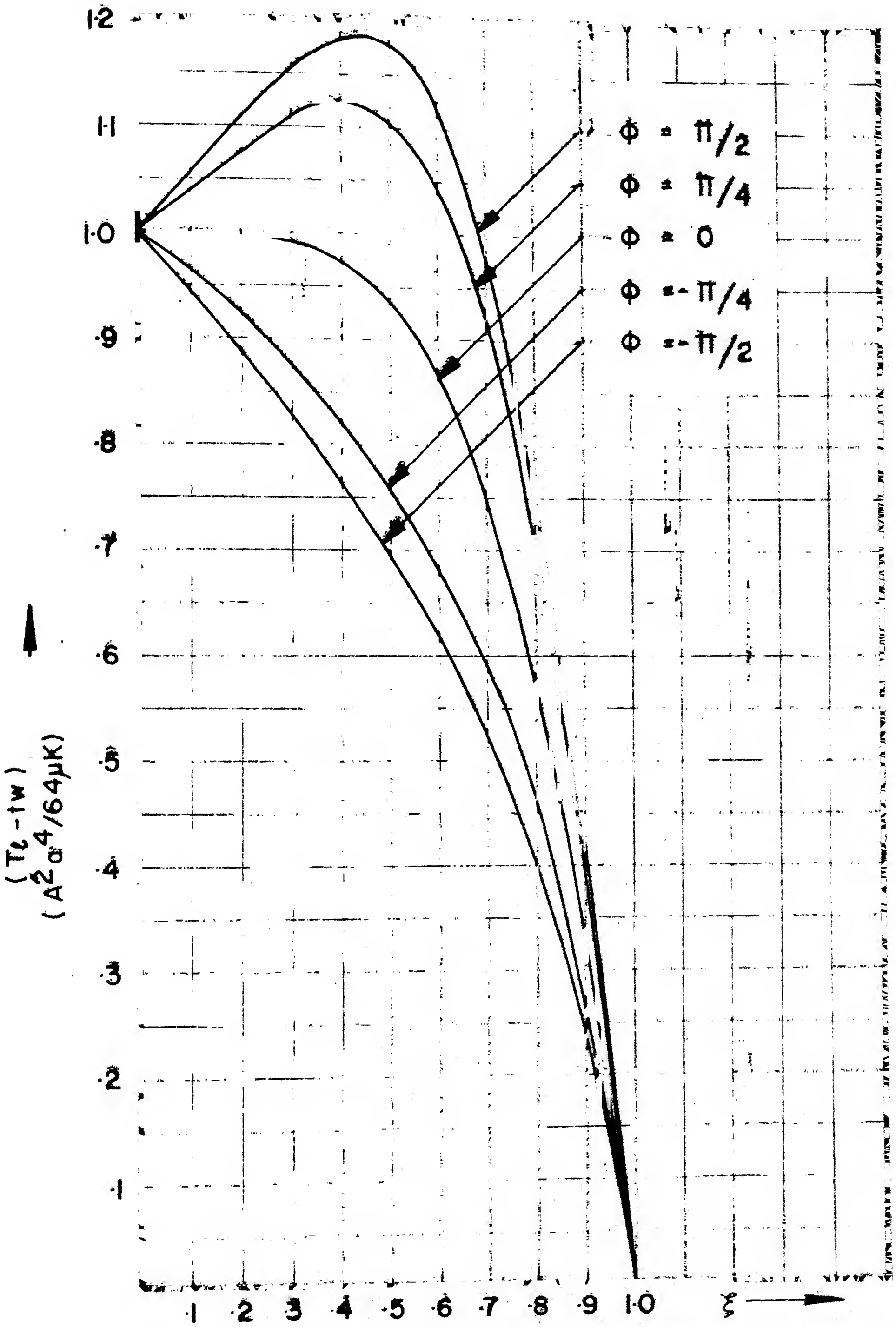

Figure 8·7- The temperature field inside the curved pipe for $Pr = 7{\cdot}5$, $R = 100$ on all the radii vectors.

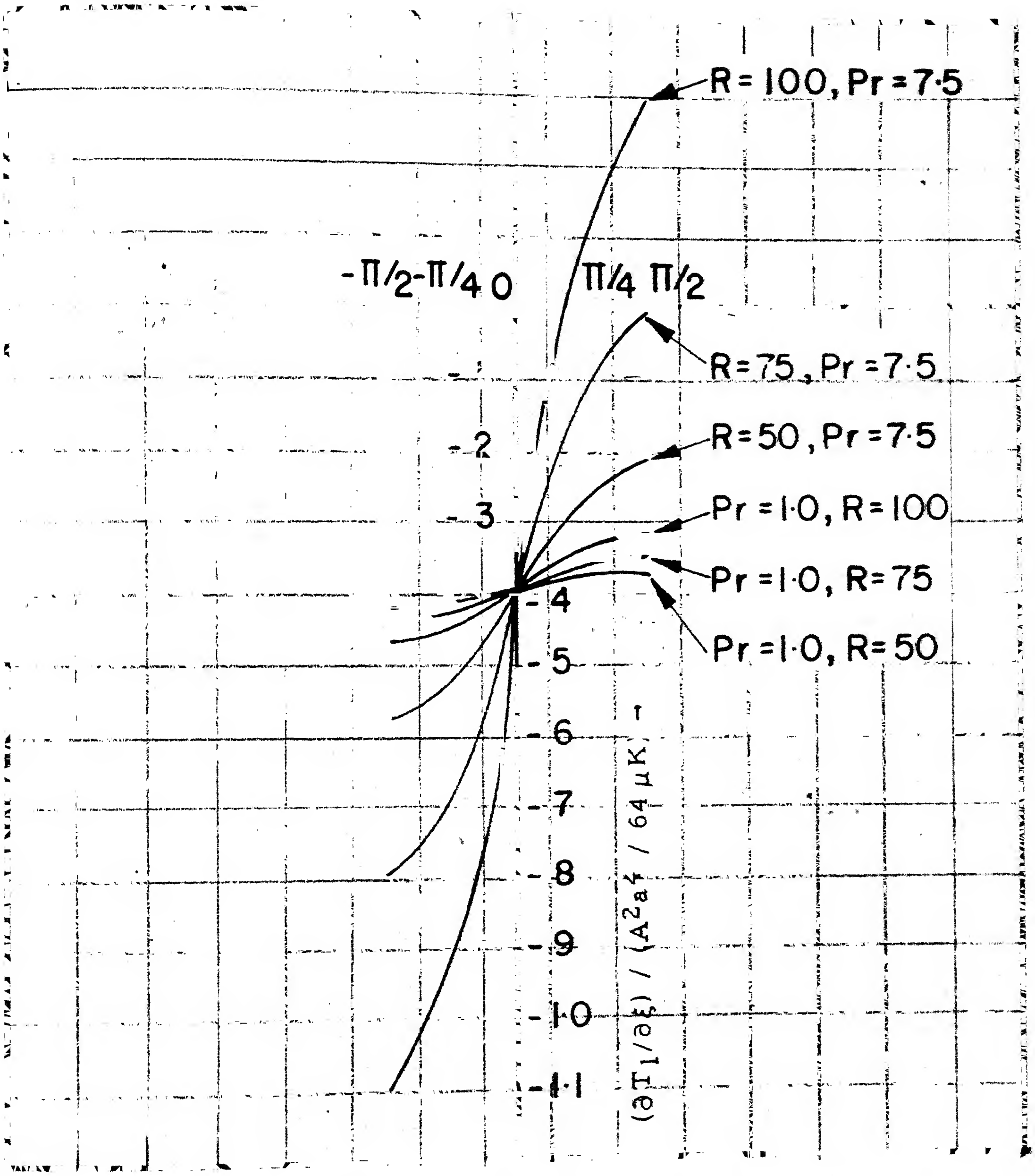

Figure.8·8- The wall-heat transfer rate of the curved pipe at various points on the cross section of the pipe for Pr = 7·5, R = 75.

(vi) Graph (8.7) shows the temperature curves for R = 100 and P_r = 7.5 on the same radii vectors as before. Here we notice that the temperature difference at about ξ = .45 still increases with the in P_r and R. On the horizontal diameter at about ξ = .55 the temperature difference is maximum on two sides of the centre of the pipe.

Since the expression for the skin friction does not involve P_r, we can say that the higher temperature difference in the liquid on both sides of the vertical diameter is more due to the value of P_r than due to the rise in skin friction, because the skin friction is higher on the convex side of the pipe than on the concave side.

(vii) Finally graph no.(8.8) shows the rate of heat-transfer on the surface of the pipe at various points on the cross-section of the pipe. We observe that for all combinations of R and P_r, the local heat-transfer rate is highest at the end of the horizontal diameter on the convex side and lowest on the other end of it. At the ends of the vertical diameter, it is constant and independent of the radius of R and P_r. The rate of heat-transfer goes on decreasing as we go on decreasing either P_r or R.

The effect of curvature on the heat transfer is larger than that on the skin friction and this is expected because the skin friction is proportional to the velocity gradient and the local heat-transfer rate is proportional to the second power of the velocity gradient.

CHAPTER IX

CONVECTIVE HEAT TRANSFER IN A CURVED ANNULUS STREAMLINE FLOW

9.1 INTRODUCTION

In the seventh chapter we extended the work of Dean (1927) to include the case of curved annulus geometry and confined ourselves to the analysis of hydrodynamic problem only. In the preceeding chapter, we extended Dean's problem to include the case of convective heat-transfer due to viscous dissipation in curved pipe flow. In the present chapter we wish to extend the study of chapter VIII to include the case of curved annulus geometry already considered in chapter VII. In other words the purpose of this chapter is to study the convective heat transfer due to viscous dissipation in the fluid motion considered in chapter VII. More specifically,

our purpose is to consider the heat generation due to viscosity in laminar, steady, constant property fluid and fully developed (both hydrodynamically and thermally) forced convection of a Newtonian liquid inside a curved annulus. The temperatures of both the inner and the outer cylinderical surfaces are constant (at the same temperature) with respect to time and space coordinates. This is only for simplicity that we have taken the temperatures of both the cylindrical surfaces to be equal. The problem of curved annulus heat transfer is of interest in double pipe heat exchangers and in nuclear reactors design.

9.2 MATHEMATICAL ANALYSIS

Here also we adopt the coordinate system given in figure (7.1) in chapter VII. The assumption that a/l is of first order of smallness (i.e. $(a/l)^2, (a/l)^3 \cdots$ are negligible as in chapter VII) is also retained for the purpose of satisfying energy equation in the following analysis.

We know that the velocity profile for a fully developed forced flow under a constant pressure gradient of a Newtonian fluid along the axis of the straight annulus is given by

$$\omega_1 = \frac{Aa^2}{4\mu}\left(1-\rho^2-\lambda \ln\rho\right) \qquad (9.2.1)$$

where $\rho = \frac{r}{a}$, $\lambda = \frac{1-\sigma^2}{\ln\sigma}$, $\sigma = \frac{b}{a}$ and A is the constant

pressure gradient along the axis of the annulus, and

$$p = -Az + B \qquad (9.2.2)$$

together with $U = 0$ and $V = 0$.

The energy equation in the case of straight pipe cited in the previous chapter holds good for straight annulus also. Solving that energy equation under the condition $t_{1l} = t_\omega$ at $\rho = 1$ and $\rho = \sigma$, we find that the temperature at any point inside the straight annulus is given by

$$t_{1l} = t_\omega + \frac{A^2 a^4}{64 \mu K}\left[1 + 4\lambda + 4D\ln\rho - 2\lambda^2(\ln\rho)^2 - 4\lambda\rho^2 - \rho^4\right] \qquad (9.2.3)$$

where $D = \frac{1}{4\ln\sigma}\left[2\lambda^2(\ln\sigma)^2 - 4\lambda(1-\sigma^2) - (1-\sigma^4)\right] \qquad (9.2.4)$

and t_ω is the temperature at both the walls.

Since we have assumed the curvature of the annulus to be very small, we can suppose

$$T = t_{1l} + t_l \qquad (9.2.5)$$

similar to what we did for velocity components and pressure in chapter VII. T, given by this equation has been written at the same level of approximation at which velocity components and pressure has been in the preceeding chapter. Here t_l is the contribution to the temperature field due to the

curvature of the annulus, and hence forth we will call this as perturbation temperature.

The physical picture of the energy equation suggest that the perturbation temperature t_ℓ (on the same lines as velocity, stream function and pressure) should have the form

$$t_\ell = \frac{A^2 a^5}{64 \mu k \ell} t'_\ell(s) \sin\phi \tag{9.2.6}$$

along with the forms of ω, ψ and p as assumed in the VII chapter.

Equation (9.2.6) gives

$$\frac{\partial^2 t_\ell}{\partial \phi^2} = -\frac{A^2 a^5}{64 \mu k \ell} t'_\ell(s) \sin\phi$$

$$= - t_\ell \tag{9.2.7}$$

Thus the energy equation (8.2.3) becomes

$$\rho C_p u \frac{dt_{1\ell}}{ds} = k \left(\frac{\partial^2 t_\ell}{\partial s^2} + \frac{1}{s}\frac{\partial t_\ell}{\partial s} + \frac{1}{s^2}\cdot\frac{\partial^2 t_\ell}{\partial \phi^2}\right)$$

$$+ k \frac{\sin\phi}{\ell} \frac{dt_{1\ell}}{ds} + 2\mu \frac{dw_1}{ds}\left(\frac{dw}{ds} - \frac{w_1 \sin\phi}{\ell}\right) \tag{9.2.8}$$

The solution of the above equation will give the temperature t_ℓ, knowing $t_{1\ell}$ from (9.2.3) for the straight annulus.

9.3 SOLUTION FOR t_ℓ'

Substituting the values of ω_1 and ω the velocities for the straight annulus and its perturbation velocity respectively for the curved annulus from chapter VII we reduce (9.2.8) to the form

$$\rho\, C_p \frac{dt_{1\ell}}{d\varsigma} = k\left(\frac{d^2 t_\ell}{d\varsigma^2} + \frac{1}{\varsigma}\frac{dt_\ell}{d\varsigma} - \frac{1}{\varsigma^2} t_\ell\right) + k\frac{\sin\phi}{\ell}\frac{dt_{1\ell}}{d\varsigma}$$

$$+ \frac{A^2 a^3}{8\mu\ell}\left(2\varsigma + \frac{\lambda}{\varsigma}\right)\left(1-\varsigma^2-\lambda\ln\varsigma\right)\sin\phi - \frac{Aa}{2}\left(2\varsigma+\frac{\lambda}{\varsigma}\right)\frac{d\omega}{d\varsigma} \tag{9.3.1}$$

which further reduces to

$$\left(\frac{d^2 t_\ell'}{d\varsigma^2} + \frac{1}{\varsigma}\frac{dt_\ell'}{d\varsigma} - \frac{1}{\varsigma^2}t_\ell'\right) = -4 P_r R\left(\frac{\psi'(\varsigma)}{\varsigma}\right)\left(\frac{\lambda^2\ln\varsigma - D}{\varsigma} + 2\lambda\varsigma + \varsigma^3\right)$$

$$+ 8\left(2\varsigma+\frac{\lambda}{\varsigma}\right)\frac{d\omega'}{d\varsigma} + \Big[20\varsigma^3 - 16(1-\lambda)\varsigma$$

$$- \frac{4D+8\lambda}{\varsigma} + \frac{12\lambda^2\ln\varsigma}{\varsigma} + 16\lambda\varsigma\ln\varsigma\Big] \tag{9.3.2}$$

where the non-dimensional quantities are defined by

$$\varsigma = r/a$$

Prandtl number $\quad P_r = \dfrac{\mu C_p}{K}$

Reynold's number $\quad R = \dfrac{\rho A a^3}{4\mu^2} \qquad (9.3.3)$

Thermal conductivity $\quad K = \dfrac{A^2 a^3}{64\mu\ell}$

The equation (9.3.2) still envolves $\psi'(\varsigma)$ and $\omega'(\varsigma)$.

and these we have already obtained in chapter VII and are given by (7.4.3) and (7.5.2) respectively.

Substituting all these values from the VII chapter we get (9.3.2) as

$$\frac{d}{d\xi}\left\{\frac{1}{\xi}\frac{d}{d\xi}(\xi t_e')\right\} = \frac{c_1}{\xi^3} + \frac{c_2}{\xi} + c_3\xi + c_4\xi^3 + c_5\xi^5$$

$$+ c_6\xi^7 + c_7\xi^9 + c_{18}\xi(\ln\xi)^3$$

$$+ \left(\frac{c_8}{\xi^3} + \frac{c_9}{\xi} + c_{10}\xi + c_{11}\xi^3 + c_{12}\xi^5 + c_{13}\xi^7\right)\ln\xi$$

$$+ \left(\frac{c_{14}}{\xi} + c_{15}\xi + c_{16}\xi^3 + c_{17}\xi^5\right)(\ln\xi)^2 \qquad (9.3.4)$$

where $c_1, c_2 \ldots\ldots c_{18}$ are given by

$$c_1 = S\,Da_1 + 8\lambda(b_{11} - b_1)$$

$$c_2 = -S(2\lambda a_1 - Da_2) + 8(2b_{11} - 2b_1 + \lambda b_2 + \lambda b_{10}) - 4D - 8\lambda$$

$$c_3 = -S(a_1 + 2\lambda a_2 - Da_3) + 8(2b_2 + 2b_{10} + 3\lambda b_3 + \lambda b_9) + 16\lambda + 16$$

$$c_4 = -S(a_2 + 2\lambda a_3 - Da_4) + 8(6b_3 + 2b_9 + 5\lambda b_4 + \lambda b_8) + 20$$

$$c_5 = -S(a_3 + 2\lambda a_4 - Da_5) + 8(10b_4 + 2b_8 + 7\lambda b_5 + \lambda b_7)$$

$$c_6 = -S(a_4 + 2\lambda a_5) + 8(14b_5 + 2b_7 + 9\lambda b_6)$$

$$c_7 = -S(a_5) + 14\,b_6$$

$$c_8 = -S(\lambda^2 a_1) - 8\lambda\,b_{11}$$

$$c_9 = -S(\lambda^2 a_2 - Da_8) + 8(\lambda b_{10} + 2\lambda b_{14} - 2b_{11}) + 12\lambda^2$$

$$c_{10} = -S(\lambda^2 a_3 - Da_7 + 2\lambda a_8) + 8(2b_{10} + 4b_{14} + 2\lambda b_{13} + 3\lambda b_9) + 16\lambda$$

$$c_{11} = -S(\lambda^2 a_4 - Da_6 + a_8 + 2\lambda a_7) + 8(6b_9 + 4b_{13} + 5\lambda b_8 + 2\lambda b_{12})$$

$$c_{12} = -S(\lambda^2 a_5 + a_7 + 2\lambda a_6) + 8(10 b_8 + 4b_{12} + 7\lambda b_7)$$

$$c_{13} = -Sa_6 + 112\, b_7$$

$$c_{14} = -S\lambda^2 a_8 + 8\lambda\, b_{14}$$

$$c_{15} = -S(\lambda^2 a_7 - D a_9) + 8(3\lambda\, b_{13} + 2b_{14})$$

$$c_{16} = -S(\lambda^2 a_6 + 2\lambda a_9) + 8(6b_{13} + 5\lambda\, b_{12})$$

$$c_{17} = -S a_9 + 80\, b_{12}$$

$$c_{18} = -S\lambda^2 a_9 \qquad (9.3.5)$$

where $S = 4P_r.R^2$ and the constants $a_1, a_2 \cdots\cdots a_9$ and $b_1, b_2 \cdots\cdots b_{14}$ are given by

$$a_1 = H/R \quad ; \quad a_6 = -\lambda/48$$

$$a_2 = (2G-F)/4R \quad ; \quad a_7 = (4\lambda + 5\lambda^2)/32$$

$$a_3 = (8E/R - 20\lambda - 17\lambda^2)/128 \quad ; \quad a_8 = F/2R$$

$$a_4 = (3+2\lambda)/144 \quad ; \quad a_9 = -3\lambda^2/16 \qquad (9.3.6)$$

$$a_5 = -1/288$$

$$b_1 = J$$

$$b_2 = (4I - R\lambda F - 2\lambda + 4RH + R\lambda G)/8$$

$$b_3 = (768 - 128\, RG + 160\, RF - 8R\lambda E + 32\, R^2\lambda^2 + 39\, R^2\lambda^3)/1024$$

$$b_4 = (468\; R^2\lambda^2 + 171\; R^2\lambda - 144\, RE)/27648$$

$$b_8 = -R^2(32\lambda + 19\lambda^2)/3072$$

$$b_9 = -R(8F + R^2\lambda^2 + 2R^2\lambda^3)/64$$

$$b_{10} = (2\lambda - 4RH - R\lambda G + R\lambda F)/4$$

$$b_{11} = R\lambda H/2$$

$$b_{12} = R^2\lambda^2/192$$

$$b_{13} = R^2\lambda^3/128$$

$$b_{14} = -R\lambda F/8 \qquad (9.3.7)$$

The final integration of equation (9.3.4) yields

$$t_\ell'(\xi) = \frac{N_2}{\xi} + \left(\frac{N_1}{2} + d_1\right)\xi + d_2\xi^3 + d_3\xi^5 + d_4\xi^7 + d_5\xi^9 + d_6\xi^{11}$$
$$+ \left(\frac{d_7}{\xi} + d_8\xi + d_9\xi^3 + d_{10}\xi^5 + d_{11}\xi^7 + d_{12}\xi^9\right)\ln\xi$$
$$+ \left(\frac{d_{13}}{\xi} + d_{14}\xi + d_{15}\xi^3 + d_{16}\xi^5 + d_{17}\xi^7\right)(\ln\xi)^2$$
$$+ \left(d_{18}\xi + d_{19}\xi^3\right)(\ln\xi)^3$$

(9.3.8)

where N_1 and N_2 are constants of integration and $d_1, d_2, \ldots\ldots d_{19}$ are given by

$$d_1 = (-2c_2 + c_9 - c_{14})/8$$

$$d_2 = (32c_3 - 24c_{10} + 28c_{15} - 45c_{18})/256$$

$$d_3 = (72c_4 - 30c_{11} + 19c_{16})/1728$$

$$d_4 = (288c_5 - 84c_{12} + 37c_{17})/13824$$

$$d_5 = (40c_6 - 9c_{13})/3200 \quad ; \quad d_{15} = (4c_{15} - 9c_{18})/32$$

$$d_6 = c_7/120 \quad ; \quad d_{16} = c_{16}/24$$

$$d_7 = -(4c_1 + c_8)/8 \quad ; \quad d_{17} = c_{17}/48$$

$$d_8 = (2c_2 - c_9 + c_{14})/4 \quad ; \quad d_{18} = c_{14}/6$$

$$d_9 = (8c_{10} - 12c_{15} + 21c_{18})/64 \quad ; \quad d_{19} = c_{18}/8$$

$$d_{10} = (6c_{11} - 5c_{16})/144$$

$$d_{11} = (12c_{12} - 7c_{17})/576$$

$$d_{12} = c_{13}/180$$

$$d_{13} = -c_8/4$$

$$d_{14} = (c_9 - c_{14})/4 \tag{9.3.9}$$

The constants of integration N_1 and N_2 are to be determined from the boundary conditions

$$t_\ell'(1) = 0$$

$$\text{and} \quad t_\ell'(\sigma) = 0 \tag{9.3.10}$$

Thus we get

$$N_2 = \frac{\sigma^2}{(1-\sigma^2)}\Big[\{(1-\sigma^2)d_2 + (1-\sigma^4)d_3 + (1-\sigma^6)d_4 + (1-\sigma^8)d_5 + (1-\sigma^{10})d_6\}$$

$$-\frac{(\ln\sigma)}{\sigma}\left\{\frac{d_7}{\sigma} + \sigma d_8 + \sigma^3 d_9 + \sigma^5 d_{10} + \sigma^7 d_{11} + \sigma^9 d_{12}\right\}$$

$$-\frac{(\ln\sigma)^2}{\sigma}\left\{\frac{d_{13}}{\sigma} + \sigma d_{14} + \sigma^3 d_{15} + \sigma^5 d_{16} + \sigma^7 d_{17}\right\}$$

$$-\frac{(\ln\sigma)^3}{\sigma}\{\sigma d_{18} + \sigma^3 d_{19}\}\Big] \tag{9.3.11}$$

and $$N_1 = -2(N_2 + d_1 + d_2 + d_3 + d_4 + d_5 + d_6) \qquad (9.3.12)$$

Thus knowing the perturbation temperature t_ℓ' with the help of equations (9.2.6), (9.3.8),(9.3.11) and (9.3.12), we get the temperature at any point on a cross-section of the curved annulus by a plane θ = constant for fully developed temperature profile case, as follows :

$$T = t_{1\ell} + t_\ell$$

$$= t_w + \frac{A^2a^4}{64\mu K}\left[1 + 4\lambda + 4D\ln\gamma - 2\lambda^2(\ln\zeta)^2 - 4\lambda\zeta^2 - \zeta^4\right]$$

$$+ \frac{A^2a^5}{64\mu K\ell}\, t_\ell'(\zeta)\,.\,\sin\phi \qquad (9.3.13)$$

which may further be written as

$$\frac{T - t_w}{(A^2a^4/(64\mu K))} = \left[1 + 4\lambda + 4D\ln\zeta - 2\lambda^2(\ln\zeta)^2 - 4\lambda\zeta^2 - \zeta^4\right]$$

$$+ \frac{a}{\ell}\,.\, t_\ell'(\zeta)\,.\,\sin\phi \qquad (9.3.14)$$

The second term on the right hand side of the above equation (9.3.14) is to be regarded as the super-imposed temperature over the temperature in a straight annulus (the first term on the R.H.S. of (9.3.14)) due to the curvature in the annulus.

This will be zero for large radius of curvature ℓ i.e. for the straight annulus.

9.4 THE HEAT-TRANSFER-RATE AT THE WALLS OF THE ANNULUS

To find the heat-transfer-rate, we differentiate (9.3.8) and get

$$\frac{dt_e'}{d\rho} = \frac{d_7 - N_2}{\rho^2} + \left(\frac{N_1}{2} + d_1 + d_8\right) + (3d_2 + d_9)\rho^2 + (5d_3 + d_{10})\rho^4$$

$$+ (7d_4 + d_{11})\rho^6 + (9d_5 + d_{12})\rho^8 + 11 d_6 \rho^{10}$$

$$+ \left\{ (2d_{13} - d_7)/\rho^2 + (d_8 + 2d_{14}) + (3d_9 + 2d_{15})\rho^2\right.$$

$$\left. + (5d_{10} + 2d_{16})\rho^4 + (7d_{11} + 2d_{17})\rho^6 + 9d_{12}\rho^8 \right\} (\ln\rho)$$

$$+ \left\{ -\frac{d_{13}}{\rho^2} + (d_{14} + 3d_{18}) + (3d_{15} + 3d_{19})\rho^2\right.$$

$$\left. + 5d_{16}\rho^4 + 7d_{17}\rho^6 \right\} (\ln\rho)^2$$

$$+ \left\{ d_{18} + 3d_{19}\rho^2 \right\} (\ln\rho)^3 \qquad (9.4.1)$$

This is regarded as the perturbation term from the heat-transfer rate over the straight annulus and we get the following expression for the heat-transfer-rate at the walls $\rho = \sigma$ and $\rho = 1$ (as the case may be) for the curved annulus

$$\frac{\frac{dT}{d\xi}}{(A^2a^4/64\mu K)} = \left[-\frac{4D}{\xi} - \frac{4\lambda^2 \ln \xi}{\xi} - 8\lambda\xi - 4\xi^3 \right]_{\xi=\sigma,\,1}$$

$$+ \frac{a}{L} \cdot \left.\frac{dt_1'}{d\xi}\right|_{\xi=\sigma,1} \cdot \sin\phi \qquad (9.4.2)$$

9.5 DISCUSSIONS

We have performed numerical computations on equation (9.3.14) on IBM 7044 Computer. The numerical values of the temperature inside the annular region are given in tables (9.1) to (9.5). Specifically these tables given the values of $(T-t_w)/(A^2a^4/64\mu k)$ at several points on both sides of the horizontal diameter. As in chapter VII the choice of R - the Reynold's number is arbitrary in the range of 100 to 500. The radii ratio of the two cylinders has been taken to vary from .1 to .5 . The Prandtl number P_r has been taken to be 1, and it is also in the range of the Newtonian liquids. The curvature of the coil has been taken to be .02 which is sufficiently small in comparison with the radii ratios. Apart from the points on the horizontal diameter we have also calculated the temperature profile on the radii vectors $\phi = 0$ and $\phi = \pm\frac{\pi}{4}$. In general we see that the contribution of the perturbation temperature t_l on the concave and convex side of the vertical diameter is just opposite in sign. Thus we discuss all the cases on only the convex side $\phi = \frac{\pi}{2}$ of the horizontal diameter.

Table (9.1) gives the temperature profile $t_{1\ell}$ for the straight annulus and the perturbation temperature t_{ℓ} over $t_{1\ell}$ assumed due to the curvature of the coil. The Reynold's number has been taken at 100 and the radius ratio .1 and the curvature at .02. We see that in case of straight annulus, the temperature rises more steeply near the inner wall than near the outer wall. In between the walls it becomes more stationary with the maximum at $\zeta = .45$. We also find that the Reynold number R has no effect on this temperature profile for the given radius ratio of the two concentric pipes. The perturbation temperature increases in the beginning upto $\zeta = .25$ and than it becomes negative till $\zeta = .40$ and then again it becomes positive and remains so till $\zeta = .75$ to become negative again till the outer wall. The net result being the resultant of the two to make the temperature more stationary in between the walls. Other things remaining the same if we increase Reynolds number to 200, the temperature profile for the straight annulus remains the same but the perburbation temperature beguns with a positive value from near the inner wall and remains so till $\zeta = .70$ and after that it becomes negative and remains so till the outer wall. This results in a steep fall near the outer wall. As we increase R the contribution of the perturbation temperature becomes more and more negative after the same point $\zeta = .75$. This tendency of the temperature profile remains at the radii vector $\alpha = \frac{\pi}{4}$ also,

becoming zero at the vertical diameter. Thus we can say that the effect of t_e is maximum at the horizontal diameter and becomes smaller and smaller as we move upwards along on arc somewhat parallel to the outer circumference reducing to a zero at the vertical diameter. This happens for all R increasing.

Table (9.2) gives the temperature profiles in the same case as above but the radii ratio increases to $\sigma = .2$. We see that for $R = 100$, the perturbation temperature is everywhere negative on the horizontal diameter on the convex side of it. When R increases to 200, all other parameters remaining the same, the temperature t_e becomes positive in the neighbourhood of the inner wall ($\zeta = .25$) and immediately afterwards it becomes negative and remains so throughout upto the outer wall. For increasing R the tendency of the point of change of sign is to move towards the outer wall reaching $\zeta = .30$ for $R = 500$. For values of $\sigma \geqslant .3$ and $R \geqslant 300$ the perturbation temperature t_e remains negative throughout the convex side of the annulus, with the result that the temperature on this side is below the wall temperature whereas on the concave side it is higher than the wall temperature. This is evident from the tables (9.3) to (9.5). Apart from the tables we have also found that no change of sign for the perturbation temperature occures for lower Reynolds number in the range of 25 to 75.

For some values of σ near zero, the perturbation temperature values are negative at some points, and positive at other points on the horizontal diameter. For larger values of σ the values of the same temperature t_2 are negative at all points of the same diameter. This further shows that for large values of σ the temperature is decreased on the convex side and increased on the concave side.

We have also computed the local heat-transfer-rate from equation (7.4.2) on the walls of the annulus for the same set of parameters as before. The points on the surfaces have been taken to be the radii vectors $\phi = 0$, $\pm \frac{\pi}{4}$, $\pm \frac{\pi}{2}$. Here we see that at the end of the vertical diameter i.e. $\phi = 0$ the perturbation temperature is always zero for all set of values of R and σ, because of the multiplication of $\sin\phi$. The perturbation temperatures at $\phi = \pm \frac{\pi}{4}$ have the same numerical value but have opposite signs. The same is the case at $\phi = \pm \frac{\pi}{2}$ hence we study the perturbation temperature only at the ends of the radii vectors $\phi = \frac{\pi}{4}$ and $\phi = \frac{\pi}{2}$.

The wall-heat-transfer-rate of the straight annulus is not affected by the Reynolds number R or the Prandtl number P_r, but the wall-heat-transfer-rate is of the curved annulus is affected by them. The table number 9.6 gives the heat-transfer-rate at both the inner and the outer walls of the curved annulus for fixed values of the radii

ratio $\sigma = .1$, the Prandtl number $P_r = 1.0$ and the curvature $\frac{a}{\ell} = .02$ but for different values of R. From this table we find that as R increases the magnitude of the secondary heat transfer rate at both the walls increases whereas the signs of the heat-transfer rate at the inner wall being negative but that on the outer wall is positive. In other words we can say that due to the curvature in the annulus the heat on the inner wall flows towards the wall and that of the outer wall moves towards the liquid. The net result is that upto $R = 300$ the total heat transfer rate on the outer wall of the annulus remains negative and after this it becomes positive there also.

The effect of σ on the wall-heat-transfer rate has been tabulated in table no. 9.7 . Here we have taken the same parameters as before for fixed Reynolds number $R = 100$ but for changing σ. We find that as σ increases, the secondary heat-transfer-rate at both the walls increases in magnitude. For $\sigma = .1$, the secondary-heat-transfer-rates at both the walls are opposite in sign, but for larger values of σ the two rates are of the same sign, that is to say it is positive on the convex side and negative on the concave side for all values of ϕ. We also find that the sign of the secondary-heat-transfer-rate changes for some values of σ in the vicinity of $\sigma = .2$. The net result in this case is that the total heat-transfer-rate

on the inner wall for all values of ϕ remains positive on the convex side whereas on the outer wall it changes from negative to positive as σ increases.

Table 9.1 : The temperature profiles on $\Phi = \frac{\pi}{2}$ for $P_r = 1$, $R = 100$, $\frac{a}{\ell} = .02$ and $\sigma = .1$.

γ	$\frac{(t_{1\ell}-t_w)}{(A^2a^4/64\mu k)}$	$\frac{t_\ell}{(A^2a^4/64\mu k)}$	$\frac{(T-t_w)}{(A^2a^4/64\mu k)} = t_{1\ell}+t_\ell$
.10	.000000	.000000	.000000
.15	.181742	-.000869	.180874
.20	.251005	-.000386	.250619
.25	.276480	+.000285	.276765
.30	.284331	.000677	.284006
.35	.282494	.000642	.282126
.40	.270931	.000210	.279240
.45	.275175	-.000490	.274935
.50	.271629	-.001260	.270369
.55	.268179	-.001840	.266319
.60	.264012	-.002086	.261926
.65	.257890	-.001563	.2[illegible]19
.70	.242254	-.000096	.242158
.75	.232981	+.002337	.235318
.80	.210923	.005275	.216198
.85	.178223	.009343	.187567
.90	.134881	.010201	.145081
.95	.076160	.008565	.084728
1.00	.000000	.000000	.000000

Table 9.2 : The temperature profiles on $\phi = \pi/2$ for $P_r = 1$, $R = 200$, $\frac{a}{\ell} = .02$ and $\sigma = .1$.

ζ	$\frac{(t_{1\ell}-t_w)}{(A^2a^4/64\mu k)}$	$\frac{t_\ell}{(A^2a^4/64\mu k)} = \frac{a}{\ell}\sin\phi .\, t_\ell'$	$\frac{(T-t_w)}{(A^2a^4/64\mu k)} = t_{1\ell}+t_\ell$
.10	.000000	.000000	.000000
.15	.181742	.009172	.190914
.20	.251005	.009299	.261005
.25	.276480	.008421	.284901
.30	.283331	.006935	.290267
.35	.282494	.006540	.289034
.40	.279031	.007434	.286465
.45	.275175	.009335	.284510
.50	.271629	.011636	.283265
.55	.268179	.013495	.281674
.60	.264012	.013918	.277930
.65	.257890	.011885	.269775
.70	.248254	.006563	.254817
.75	.233281	-.002364	.230917
.80	.210923	-.014215	.196708
.85	.178933	-.031651	.147282
.90	.134881	-.034866	.100914
.95	.076160	-.030417	.045743
1.00	.000000	.000000	.000000

Table 9.3 : The temperature profiles on $\phi = \pi/2$ for $P_r = 1.0$, $R = 300$, $\frac{a}{\ell} = .02$ and $= .1$

ζ	$\frac{(t_{1\ell} - t_w)}{(A^2a^4/64\mu k)}$	$\frac{t_\ell}{(A^2a^4/64\mu k)} = \frac{a}{\ell} \cdot \sin\phi \cdot t_\ell'$	$\frac{(T - t_w)}{(A^2a^4/64\mu k)} = t_{1\ell} + t_\ell$
.10	.000000	.000000	.000000
.15	.181742	.023009	.204751
.20	.251005	.026019	.277025
.25	.276480	.022929	.299409
.30	.283331	.019620	.302952
.35	.282494	.018510	.301004
.40	.279031	.020173	.299204
.45	.275175	.024076	.299251
.50	.271629	.028929	.300558
.55	.268179	.032886	.301065
.60	.264012	.033738	.297750
.65	.257890	.029190	.287080
.70	.248254	.017341	.265595
.75	.233281	-.002575	.230706
.80	.210923	-.029116	.181807
.85	.178933	-.057158	.121776
.90	.134881	-.075875	.059006
.95	.076160	-.066930	.009230
1.00	.000000	.000000	.000000

Table 9.4 : The temperature profiles on $\phi = \frac{\pi}{2}$ for P_r = 1.00, R = 400, $\frac{a}{l}$ = .02 and σ = .2 .

ρ	$\dfrac{(t_{1e} - t_w)}{(A^2a^4/64\mu k)}$	$\dfrac{t_e}{(A^2a^4/64\mu k)} = \frac{a}{l} \cdot \sin\phi \cdot t_e'$	$\dfrac{(T - t_w)}{(A^2a^4/64\mu k)} = t_{1e} + t_e$
.20	.000000	.000000	.000000
.25	.092355	.024082	.116436
.30	.134652	.001030	.135682
.35	.158213	-.038441	.119772
.40	.157781	-.059031	.098750
.45	.158048	-.072700	.085348
.50	.156470	-.074370	.082100
.55	.154610	-.069274	.085336
.60	.152847	-.065059	.087788
.65	.150764	-.070110	.080654
.70	.147378	-.091669	.055709
.75	.141283	-.133420	.007863
.80	.130733	-.192189	-.061456
.85	.113702	-.243477	-.129775
.90	.087924	-.285509	-.197585
.95	.050916	-.231612	-.180696
1.00	.000000	.000000	.000069

Table 9.5 : The temperature field on $\phi = \pi/2$ for P_r = 1.0, R = 500, $\frac{a}{\ell}$ = .02 and σ = .2 .

ζ	$\dfrac{(t_{1\ell} - t_w)}{(A^2 a^4/64\mu k)}$	$\dfrac{t_\ell}{(A^2 a^4/64\mu k)} = \frac{a}{\ell} . \sin\phi . t_\ell'$	$\dfrac{(T - t_w)}{(A^2 a^4/64\mu k)} = t_{1\ell} + t_\ell$
.20	.000000	.000000	.000000
.25	.092355	.038473	.130828
.30	.134652	.002904	.137557
.35	.152213	-.049197	.102017
.40	.157781	-.090707	.067073
.45	.158048	-.112126	.045922
.50	.156470	-.114839	.041631
.55	.154610	-.106926	.047684
.60	.152847	-.100491	.052356
.65	.150764	-.108439	.042324
.70	.147378	-.142148	.005231
.75	.141983	-.207362	[illegible]
.80	.130733	-.299211	-.168478
.85	.113702	-.395019	-.281317
.90	.087924	-.448213	-.367236
.95	.050916	-.361176	.310259
1.00	.000000	.000000	.096609

Table 9.6 : The wall-heat-transfer-rate for radius ratio $\sigma = .1$, $P_r = 1.0$

	R	INNER WALL $\dfrac{\partial t_e/\partial \rho}{(A^2a^4/64\mu k)} = \dfrac{a}{l}\cdot \sin\phi \cdot \dfrac{\partial t_e'}{\partial \rho}$	INNER WALL $\dfrac{\partial T/\partial \rho}{(A^2a^4/64\mu k)}$	OUTER WALL $\dfrac{\partial t_e/\partial \rho}{(A^2a^4/64\mu k)} = \dfrac{a}{l}\cdot \sin\phi \cdot \dfrac{\partial t_e'}{\partial \rho}$	OUTER WALL $\dfrac{\partial T/\partial \rho}{(A^2a^4/64\mu k)}$
$\phi = \frac{\pi}{4}$	100	-.230712	5.570467	.192055	-1.554309
	200	-.569660	5.231519	.698190	-1.012694
	300	-1.134293	4.665987	1.641732	-.175142
	400	-1.925453	3.875726	2.722653	+1.005816
	500	-2.940902	2.26[illegible]	4.241074	2.684200
$\phi = \frac{\pi}{2}$	100	-.326277	5.474902	.271621	-1.44[illegible]
	200	-.805622	4.995555	.[illegible]7390	-0.729484
	300	-1.604132	4.157047	2.1[illegible]236	.463464
	400	-2.723002	3.078177	3.850465	2.182501
	500	-4.159064	1.642115	5.937785	4.260911

Table 9.7 : The wall-heat-transfer-rate for R = 100, and P_r = 1.0

	σ	INNER WALL: $\dfrac{(\partial t_e/\partial \xi)}{(A^2a^4/64\mu k)} = \frac{a}{\ell}\cdot \sin\phi \cdot \frac{\partial t_e'}{\partial \xi}$	INNER WALL: $\dfrac{(\partial T/\partial \xi)}{(A^2a^4/64\mu k)}$	OUTER WALL: $\dfrac{(\partial t_e/\partial \xi)}{(A^2a^4/64\mu k)} = \frac{a}{\ell}\cdot \sin\phi \cdot \frac{\partial t_e'}{\partial \xi}$	OUTER WALL: $\dfrac{(\partial T/\partial \xi)}{(A^2a^4/64\mu k)}$
$\phi = \frac{\pi}{4}$	.1	.230712	5.570467	.192065	-1.524809
	.2	1.272426	3.909918	.337246	[illegible]
	.3	7.164614	8.585096	.480760	-.309462
	.4	17.690296	18.472601	[illegible]	.115783
	.5	31.953809	32.365951	.734048	[illegible]
$\phi = \frac{\pi}{2}$	.1	-.326277	5.474903	.271621	[illegible]
	.2	1.799480	4.432973	.477080	-.369123
	.3	10.132294	11.552776	.679910	-.110820
	.4	25.017860	25.800163	.877136	[illegible]
	.5	45.188237	45.601279	1.022101	.7404?3

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